Couples

Two equal and opposite non-collinear forces.

Recall: \( \overrightarrow{r}_{B/A} = \overrightarrow{r}_B - \overrightarrow{r}_A \)

What is the moment due to the forces about some arbitrary point \( A \)?

\[
\overrightarrow{M}_+ = \overrightarrow{r}_{B/A} \times \overrightarrow{F}
\]

\[
\overrightarrow{M}_- = \overrightarrow{r}_{C/A} \times (-\overrightarrow{F})
\]

\[
\overrightarrow{C}_A = \overrightarrow{M}_+ + \overrightarrow{M}_- = \overrightarrow{r}_{B/A} \times \overrightarrow{F} - \overrightarrow{r}_{C/A} \times \overrightarrow{F}
\]

\[
= (\overrightarrow{r}_{B/A} - \overrightarrow{r}_{C/A}) \times \overrightarrow{F}
\]

\[
= (\overrightarrow{r}_B - \overrightarrow{r}_A - \overrightarrow{r}_C + \overrightarrow{r}_A) \times \overrightarrow{F}
\]

\[
= (\overrightarrow{r}_B - \overrightarrow{r}_c) \times \overrightarrow{F}
\]

Note these expressions do not depend on \( \overrightarrow{r}_A \) in any way!
The moment due to a couple is the same no matter what points the moments are taken about.

Note, furthermore that the sum of the forces due to a couple are zero.

i.e. \[ \mathbf{\hat{R}} = \mathbf{\hat{F}} + \mathbf{\hat{-F}} = \mathbf{0} \]

**2-D Scalar Representation**

\[ \mathbf{\hat{M}}_{A}^{+} = -F(d+l) \mathbf{\hat{k}} \]

\[ \mathbf{\hat{M}}_{A}^{-} = F(l) \mathbf{\hat{k}} \]

\[ \mathbf{\hat{C}} = \mathbf{\hat{M}}_{A}^{+} + \mathbf{\hat{M}}_{A}^{-} = -F d \mathbf{\hat{k}} \]

(Note the force couple causes a clockwise \( \mathbf{\hat{k}} \) rotation)

i.e. \( -\mathbf{\hat{k}} \) rotation
Again \( \vec{C} \) did not depend on the location of \( A \) in any way, hence the couple about any point is always the same.

This also means that conceptually you can move a couple anywhere in space and it will have the same effects on a rigid body.

**Equivalent Couples**

\[
\begin{align*}
\text{100N} & & \text{50N} \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
1\text{cm} & & 2\text{cm}
\end{align*}
\]

\[
\begin{align*}
\text{50N} & & \text{100N} \\
\rightarrow & & \rightarrow \\
2\text{cm} & & 1\text{cm}
\end{align*}
\]

Sometimes we also use the "moment arrow" i.e. \( \rightarrow \) for 3-D drawings.
Example Problem 2.75

You have 2 bolts to place in 2 of the 4 holes. Where should you place the bolts to minimize the shearing force on them? What is the max C if the max shearing force on the bolts can be 40 N?

\[ \sum \mathbf{M} = -C \mathbf{k} + Fd \mathbf{k} \]

where \( Fd \) is the moment applied by the bolts.

\[ F = \frac{C}{d} \rightarrow \text{to minimize } F \text{ we must maximize } d. \]

Therefore we want to place bolts in holes A and D.

The distance \( d \) between A and D is \( \sqrt{120^2 + 90^2} = 150 \text{ mm} \).

\[ C_{\max} = F_{\max} \cdot d = 40 \text{ N} \cdot 150 \text{ mm} = 6 \text{ N} \cdot \text{m} \]
Example Problem 2.79

For clarity I have drawn the robot arm in the $\hat{z} - \hat{e}_{40}$ plane and this is the plane of the page. The $\hat{e}_{40}$ direction is in the $x-y$ plane.

Determine the vector form of the couple and the moment of the couple about the $z$-axis.

$$\vec{C} = C \, \hat{e}_c$$

$C =$ magnitude of the couple  
$\hat{e}_c =$ direction (unit vector)

In the $z - \hat{e}_{40}$ plane we have

$$\hat{e}_c = \left( \hat{e}_c \cdot \hat{e}_{40} \right) \hat{e}_{40} + \left( \hat{e}_c \cdot \hat{k} \right) \hat{k}$$

$= \sin 30^\circ$ 
$= \cos 30^\circ$

$\vec{e}_c =$ minus b/c $\hat{e}_c$ opposes $\hat{e}_{40}$

$$\therefore \hat{e}_c = -\frac{1}{2} \hat{e}_{40} + \frac{\sqrt{3}}{2} \hat{k}$$
Now consider \( \vec{e}_{40} \) in the x-y plane.

\[
\vec{e}_{40} = (\vec{e}_{40} \cdot \hat{z}) \hat{z} + (\vec{e}_{40} \cdot \hat{y}) \hat{y} \\
= \sin 40^\circ \hat{z} - \cos 40^\circ \hat{y}
\]

\[
\vec{e}_c = \frac{1}{2} \sin 40^\circ \hat{z} - \frac{1}{2} \cos 40^\circ \hat{y} + \frac{\sqrt{3}}{2} \hat{k}
\]

\[
\vec{C} = 52 \text{ lb ft} \left( \frac{1}{2} \sin 40^\circ \hat{z} - \frac{1}{2} \cos 40^\circ \hat{y} + \frac{\sqrt{3}}{2} \hat{k} \right)
\]

Moment due to the couple about the z-axis.

Recall \( \overrightarrow{M}_{\text{axis}} = (\overrightarrow{M}_{\text{pt}} \cdot \hat{e}_{\text{axis}}) \hat{e}_{\text{axis}} \)

where \( \overrightarrow{M}_{\text{pt}} \) is the moment about any point on the axis. But, the moment due to \( \vec{C} \) about every point is just \( \vec{C} \).

\[
\overrightarrow{M}_{z-\text{axis}} = (\vec{C} \cdot \hat{k}) \hat{k} \\
= 52 \left( \frac{\sqrt{3}}{2} \right) \hat{k} \text{ ft lb}
\]
Equivalent Systems

Two sets of force systems are equivalent (as far as their effects on a rigid body are concerned) if the sum of all of the forces and the sum of the moments about any point in space are identical.

\[ \sum \vec{F} = \sum_{i=1}^{N} \vec{F}^{(i)} \]

\[ \sum \vec{M}_A = \sum_{i=1}^{M} \vec{C}^{(i)} + \sum_{i=1}^{N} \vec{r}_{i/A} \times \vec{F}^{(i)} \]

where there are \( N \) forces acting at the positions \( \vec{r}_{i} \) and there are \( M \) arbitrarily positioned couples.

Recall that the net force of a couple is zero and the moment due to a couple about every point in space is identical. Hence, a couple can be moved to any point in space without changing its effects on a rigid body.

However, changing the position of a force does change its moment about points in space.

\[ \vec{F} \]

A, B, C
For example, if I move the force to point B its moment about point C decreases and its moment about point B vanishes.

So if we insist on moving a force we must compensate for this change in moment. Here is how we do this.

\[ \vec{F} \]

We want to move \( \vec{F} \) from A to B.

At point B, place both \( \vec{F} \) and its opposite \( -\vec{F} \).

Now the \( \vec{F} \) and \( -\vec{F} \) surrounded by the dotted line constitute a couple \( \vec{C} = \vec{r}_{A/B} \times \vec{F} \).

\[ \vec{C} \]

What's left is \( \vec{F} \) acting at point B and the couple \( \vec{C} \) floating around, i.e. acting at any point you like.
Example Problem 2.91

\[ \vec{F} = 2500\hat{z} + 4000\hat{j} + 3000\hat{k} \text{ lb.} \]

Move \( \vec{F} \) to \( 0 \).

Resolve this force into a normal and shearing component.

Resolve the couple into a twisting and bending component.

By moving \( \vec{F} \) to 0 we must also add a couple \( \vec{C} \) to compensate for the change in moment.

\[ \vec{C} = \vec{r}_{A/0} \times \vec{F} \]

\[ \vec{r}_{A/0} = \vec{r}_A = 10\hat{z} + 5\hat{j} - 4\hat{k} \text{ in.} \]

\[ \vec{r}_{A/0} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 5 & -4 \\ 2500 & 4000 & 3000 \end{vmatrix} = 31000\hat{z} - 40000\hat{j} + 27500\hat{k} \text{ lb in.} \]

\[ \therefore \vec{C} = 31000\hat{z} - 40000\hat{j} + 27500\hat{k} \text{ in. lb} \]

The normal force is that parallel to the y-axis, i.e. \( \vec{F}_N = 4000\hat{j} \text{ lbs.} \).
Then the shearing part of the force is the remainder.

\[ \vec{F}_s = \vec{F} - \vec{F}_N = 2500\hat{c} + 3000\hat{k} \text{ lbs.} \]

The twisting component of \( \vec{C} \) is parallel to \( \vec{j} \), i.e.

\[ \vec{C}_T = (\vec{C} \cdot \vec{j})\vec{j} = -40000\hat{j} \text{ in lb} \]

The remainder is the bending component of \( \vec{C} \), i.e.

\[ \vec{C}_B = \vec{C} - \vec{C}_T = 3000\hat{c} + 27500\hat{k} \text{ in lb.} \]