Problem 1

The closed loop transfer function can be written as follows:

\[
G_{CL}(s) = \frac{\bar{y}(s)}{\bar{y}_{sp}(s)} = \frac{8}{(s+1)(2s+1)} = \frac{8}{2s^2 + 3s + 9} = \frac{8/9}{2/9s^2 + 1/3s + 1} \tag{S1.1}
\]

Thus, comparing (S1.1) to the standard form of 2nd order systems, one obtains:

\[
\tau^2 = \frac{2}{9} \quad \Rightarrow \quad \tau = \sqrt{\frac{2}{3}} = 0.471 \tag{S1.2}
\]

\[
2\xi\tau = 1/2 \quad \Rightarrow \quad \xi = \sqrt{\frac{2}{4}} = 0.354 \tag{S1.3}
\]

\[
k = \frac{8}{9} = 0.889 \tag{S1.4}
\]

Since \(\xi\) is smaller than 1, the system is underdamped.

a) From textbook, we know that:

\[
OS = \exp\left(\frac{-\pi\xi}{\sqrt{1 - \xi^2}}\right) = 0.305 = \frac{\text{max dev final value}}{\text{final value}} = \frac{y_{\text{max}} - y(\infty)}{y(\infty)} \tag{S1.5}
\]

The output of the system for a step change of magnitude 0.1 in the set point is:

\[
\bar{y}(s) = \frac{0.1}{s} \cdot \frac{8/9}{2/9s^2 + 1/3s + 1} \tag{S1.6}
\]

Applying the final value theorem, yields:

\[
y(\infty) = \lim_{s \to 0} s\bar{y}(s) = \lim_{s \to 0} \frac{0.8/9}{2/9s^2 + 1/3s + 1} = \frac{0.8}{9} = 0.0889 \tag{S1.7}
\]

Hence, the maximum value of the response is:

\[
y_{\text{max}} = y(\infty)(1 + OS) = 0.116 \tag{S1.8}
\]
b) For a servo problem, the offset is given by:

\[
\text{offset} = \text{new set point} - y(\infty) = 0.1 - 0.0899 = 0.0111
\]  

(S1.9)

c) From the textbook, the period of the oscillation is:

\[
T = \frac{2\pi}{\omega} \quad \text{where} \quad \omega = \sqrt{1 - \xi^2} / \tau
\]  

(S1.10)

Therefore, the value of the period of the oscillation \(T\) is:

\[
T = 3.17 \text{ min}
\]  

(S1.11)

**Problem 2**

The resistance of a liquid to a hydrostatic pressure can be defined as the rate of changing of the liquid level due to the change of the output flow rate. Thus:

\[
R = \frac{dh}{dq}
\]  

(S2.1)

Assuming linear resistances, \(R_1\) and \(R_2\) can be directly evaluated by plotting \(h\) (ft) versus \(q\) (ft\(^3\)/min) and estimating the slope of the straight line. Hence, one obtains:

\[
R_1 = R_2 = 0.5
\]  

(S2.2)

Since the cross sections are equal to 2, the time constant of the two tanks are both equal to \(\tau_1 = A_1 R_1 = 1\) (min) and \(\tau_2 = A_2 R_2 = 1\) (min), while the static gains of two tanks are equal to \(k_1 = R_1 = 0.5\) (min/ft\(^2\)) and \(k_2 = R_2 / R_1 = 1\) (min/ft\(^2\)). Thus, the transfer functions of the two tanks are the following:

\[
G_1(s) = \frac{0.5}{s+1} \quad \Rightarrow \quad G_2(s) = \frac{1}{s+1}
\]  

(S2.3)

Employing a proportional controller, the corresponding transfer function is:

\[
G_c(s) = k_c
\]  

(S2.4)

Plotting the change in pressure to the valve versus the change in flow provides the transfer function for the final control element:

\[
G_f(s) = \frac{dP}{dq} = \frac{0.1}{1} = 0.1
\]  

(S2.5)

With no lag in the measuring device dynamics, the corresponding transfer function is:
\( G_m(s) = 1 \) \hspace{1cm} (S2.6)

Therefore, the block diagram is the following:

a) The close loop transfer function is:

\[
\frac{\bar{y}(s)}{y(s)} = \frac{0.05 k_c}{(s + 1)^2} = \frac{0.05 k_c}{1 + 0.05 k_c} \]

Comparing the closed loop transfer function to the standard form of a 2\textsuperscript{nd} order system transfer function one obtains:

\[
\tau = \frac{1}{\sqrt{1 + 0.05 k_c}} \hspace{1cm} (S2.8)
\]

\[
\xi = \frac{1}{\sqrt{1 + 0.05 k_c}} \hspace{1cm} (S2.9)
\]

\[
k = \frac{0.05 k_c}{(1 + 0.05 k_c)} \hspace{1cm} (S2.10)
\]

For a critically damped 2\textsuperscript{nd} order system, \( \xi = 1 \) and therefore:

\[
k_{c(CD)} = 0 \hspace{1cm} (S2.11)
\]

Therefore, the critical damping cannot occur.

c) For interacting tanks, the transfer function between the output of the second tank and the input of the first tank is given by:

\[
G(s) = \frac{\bar{y}(s)}{\bar{q}(s)} = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + A_1 R_2) s + 1} = \frac{0.5}{s^2 + 3s + 1} \]

Thus, the close loop transfer function becomes:
Comparing the closed loop transfer function to the standard form of a 2\textsuperscript{nd} order system transfer function one obtains:

\[
\tau = 1/\sqrt{1 + 0.05 k_c} \quad \text{(S2.14)}
\]

\[
\xi = \frac{3}{2\sqrt{1 + 0.05 k_c}} \quad \text{(S2.15)}
\]

\[
k = 0.05 k_c / (1 + 0.05 k_c) \quad \text{(S2.16)}
\]

For a critically damped 2\textsuperscript{nd} order system, \(\xi = 1\) and therefore:

\[
k_{c(CD)} = 25 \text{ psi/ft} \quad \text{(S2.17)}
\]

Hence, for interacting capacities the critical damping does occur.

d) Assuming \(k_c = 1.5k_{c(CD)}\), one obtains:

\[
k_c = 1.5k_{c(CD)} = 37.5 \text{ psi/ft} \quad \text{(S2.18)}
\]

Thus, the natural period of the oscillations, damping factor and gain becomes:

\[
\tau = 0.590 \text{ (min)} \quad \text{(S2.19)}
\]

\[
\xi = 0.885 \quad \text{(S2.20)}
\]

\[
k = 0.652 \text{ (ft)} \quad \text{(S2.21)}
\]

For a step change of 1/12 ft in the se point, \(y_{sp}(s)\) is equal to \(1/(s + 12)\). Therefore, one obtains:

\[
\overline{y}(s) = \frac{0.652}{0.348s^2 + 1.043s + 1} \overline{y}_{sp}(s) = \frac{0.054}{(0.348s^2 + 1.043s + 1)s} \quad \text{(S2.22)}
\]

Since \(\xi\) is smaller than 1, from textbook we have:

\[
y(t) = 0.054 \left[ 1 - \frac{1}{\sqrt{1 - \xi^2}} e^{-\frac{t}{\tau_1}} \sin \left( \sqrt{1 - \xi^2} \frac{t}{\tau_1} + \tan^{-1} \left( \frac{\sqrt{1 - \xi^2}}{\xi} \right) \right] \right] = 0.054 - 0.116e^{-1.5t} \sin(0.8t + 28^\circ) \quad \text{(S2.23)}
\]
**Problem 3:**

a) The block diagram is the following:

![Block Diagram](image)

b) Controller (pure PID)

Transfer function is: \( G_c(s) = K_c \left( 1 + \frac{1}{\tau_i s} + \tau_d s \right) \)

Transducer

Transfer Function is: \( G_T(s) = \frac{p_v(s)}{p(s)} = \frac{p_v(s) - 3}{p(s) - 4} = \frac{15 - 3}{20 - 4} = \frac{12}{16} = \frac{3}{4} \)

Control Valve

Linearizing \( q_A \) around an operative point:

\[ q_A = q_{A_0} + 0.03 \ln(20)20^{(p_v - 3)/12} (p_v - p_{v_0}) \]

Choosing \( q_{A_0} = 0.17 \) and \( p_{v_0} = 3 \), gives: \( q_A = q_{A_0} + 0.0025 \ln(20)(p_v - p_{v_0}) \)

Transfer Function is: \( G_v(s) = \frac{\tilde{q}_A(s)}{\tilde{p}_v(s)} = \frac{q_A(s) - 0.17}{p_v(s) - 3} = 0.0025 \ln(20) \)

Also, considering delay: \( G_v(s) = 0.0025 \ln(20)e^{-s} \)

Process

\[ V \frac{dc}{dt} = q_A c_A + q_F c_F - (q_A + q_F)c \]. Note \( q_A + q_F \approx q_F \).

Hence \( V \frac{dc}{dt} + q_F c = q_A c_A + q_F c_F \).

At steady state: \( q_Fc = q_{A_0}c_A + q_P c_F \)

Defining \( \tilde{c} = c - c_s \), \( \tilde{q}_A = q_A - q_{A_s} \), \( \tilde{c}_F = c_F - c_{F_s} \) yields:

\[ V \frac{d\tilde{c}}{dt} + q_F\tilde{c} = c_A \tilde{q}_A + q_F \tilde{c}_F \] or \( \tau_p \frac{d\tilde{c}}{dt} + \tilde{c} = k_p \tilde{q}_A + k_d \tilde{c}_F \) where \( \tau_p = V/q_F \), \( K_p = c_A / q_F \) and \( K_d = 1 \).

Taking Laplace transform gives:

\[ c(s) = \frac{k_p}{\tau_p s + 1} q_A(s) + \frac{k_d}{\tau_p s + 1} c_F(s) = G_p(s)q_A(s) + G_D(s)c_F(s) \]
Therefore, process can be expanded as illustrated in figure.

Transmission Line

Transfer function is: \( G_L(s) = e^{-t_L s} \),
where \( t_L = \left[ 20\pi(0.5)^2 \right] / 4q_f \)

Composition Transmitter Data

Transfer Function is: \( G_{CTA}(s) = \frac{\tilde{c}_m(s)}{\tilde{c}(s)} = \frac{c_m(s) - 4}{c(s)} = \frac{20 - 4}{200} = \frac{16}{200} \)