Problem 1

From table 7.1 in the textbook, the Laplace transform of \( y(t) = t e^{-t} \) is:

\[
L \left[ t e^{-t} \right] = \mathcal{Y}(s) = \frac{1}{(s+1)^2} \tag{S1.1}
\]

Moreover, the Laplace transform of the unit-impulse \( \delta(t) \) is:

\[
L \left[ \delta(t) \right] = \mathcal{U}(s) = 1 \tag{S1.2}
\]

Since \( u(t) = 0 \) all the time but at \( t = t_0 \), assuming that \( y(t) \) is in deviation form (i.e. \( y_s = 0 \)), the transfer function of the system in exam can be written as follows:

\[
G(s) = \frac{\mathcal{Y}(s)}{\mathcal{U}(s)} = \frac{1}{(s+1)^2} \tag{S1.3}
\]

Problem 2

Assuming constant density and reactor volume, the overall material balance can be written as follows:

\[
A \frac{dh(t)}{dt} = F_i(t) - F = F_i(t) - 8h(t)^{1/2} \tag{S2.1}
\]

As initial condition for equation (S2.1), we chose the steady state value of the hydrostatic pressure \( h_s \) given by:

\[
h(0) = h_s = F_{i_s}^2 / 64 \tag{S2.2}
\]

Linearizing the right hand side of equation (S2.1) about the steady state value (S2.2), one obtains:

\[
F_i(t) - 8h(t)^{1/2} \approx \left[ F_i(t) - F_s \right] - \left[ 4h_s^{-1/2} \right] [h(t) - h_s] \tag{S2.3}
\]

Defining the deviation variables \( H(t) = h(t) - h_s \) and \( Q(t) = F_i(t) - F_s \), and considering (S2.3), equation (S2.2) becomes

\[
\frac{Ah_s^{1/2}}{4} \frac{dH(t)}{dt} + H(t) = \frac{h_s^{1/2}}{4} Q(t) \tag{S2.4}
\]

subject to the following initial condition

\[
H(0) = 0 \tag{S2.5}
\]

Therefore, by comparison to the standard form of 1st order systems, the time constant \( \tau \) and the gain \( k \) are the following:
\[ \tau = \frac{Ah_s^{1/2}}{4} \quad k = \frac{h_s^{1/2}}{4} \]  

(S2.6)

Hence:

a) \( h_s = 3\text{ft} \implies \tau = \frac{3\sqrt{3}}{4} \)  

b) \( h_s = 9\text{ft} \implies \tau = \frac{9}{4} \)  

(S2.7)

**Problem 3**

The material balance on the component A is the following:

\[ V \frac{dc_A(t)}{dt} = F[c_{A_i}(t) - c_A(t)] - kc_A(t)V \]  

(S3.1)

subject to the initial condition \( c_A(0) = c_{ASS} = F/(F+kV)c_{Ai,ss} \). As the differential equation describing the dynamics of the system is linear, it is already in deviation form when one chooses the deviation variables \( x(t) = c_A(t) - c_{ASS} \) and \( y(t) = c_{Ai}(t) - c_{Ai,ss} \). Thus, (S3.1) can be rewritten as follows:

\[ \frac{V}{F+kV} \frac{dx(t)}{dt} + x(t) = \frac{F}{F+kV} y(t) \]  

(S3.2)

subject to the initial condition \( x(0) = 0 \). Therefore, by comparison to the standard form of 1st order systems, the time constant \( \tau \) and the gain \( k \) of the system are given by:

\[ \tau = \frac{V}{F+kV} \quad k = \frac{F}{F+kV} \]  

(S3.3)

and the transfer function \( G(s) \) can be expressed as follows:

\[ G(s) = \frac{x(s)}{y(s)} = \frac{F}{F+kV} \frac{V}{s+1/F+kV} = \frac{k}{s+1/\tau} \]  

(S3.4)

Unit-step change \( \bar{y}(s) = 1/s \). Therefore

\[ \bar{x}(s) = \frac{k/\tau}{[s+1/\tau]s} = k \left( \frac{1}{s} - \frac{1}{s+1/\tau} \right) \]  

(S3.5)

which yields:

\[ x(t) = k(1 - e^{-t/\tau}) \]  

(S3.6)

Thus, the sketch to a unit-step change is shown in figure S3.a
**Unit-impulse change** \( \bar{y}(s) = 1 \). Therefore

\[
\bar{x}(s) = \frac{k}{\tau} \frac{1}{s + 1/\tau} \quad \text{(S3.7)}
\]

which yields:

\[
x(t) = -\frac{k}{\tau} e^{-t/\tau} \quad \text{(S3.8)}
\]

Thus, the sketch to a unit-impulse change is shown in figure S3.b

**Sinusoidal input** \( \bar{y}(s) = A\omega/(s^2 + \omega^2) \). Therefore

\[
\bar{x}(s) = \frac{k}{\tau} \frac{A\omega}{s + 1/\tau} \frac{1}{s^2 + \omega^2} \quad \text{(S3.9)}
\]

Repeating what we did in class (see also page 318 of the textbook), one gets:

\[
x(t) = \frac{kA}{\sqrt{\tau^2 \omega^2 + 1}} \sin(\omega t + \phi) \quad \phi = \tan^{-1}(-\omega t) \quad \text{(S3.10)}
\]

Thus, the sketch to a sinusoidal input is shown in figure S3.c

---

**Problem 4**

Assuming constant density and reactor volume, the overall material balance can be written as follows:

\[
AR \frac{dh(t)}{dt} + h(t) = RF(t) \quad \text{(S4.1)}
\]

As initial condition for equation (S4.1), we chose the steady state value of the hydrostatic pressure \( h_s \) given by:
Since $(S4.1)$ is linear, defining the deviation variables $H(t) = h(t) - h_s$ and $Q(t) = F(t) - F_s$ $(S4.1)$ can be immediately rewritten in deviation form as follows:

$$AR \frac{dH(t)}{dt} + H(t) = RQ(t) = Ra \sin(\omega t)$$

(S4.3)

subject to the initial condition

$$H(0) = 0$$

(S4.4)

The first order system given by $(S4.3)$-$(S4.4)$ has time constant and gain respectively given by:

$$\tau = AR \quad k = aR$$

(S4.5)

We know from what we did in class that the output of a first order system subject to a sinusoidal input is a sinusoidal wave with the same frequency and with an amplitude ratio given by:

$$ARatio = b = \frac{k}{\sqrt{\tau^2 \omega^2 + 1}} = \frac{aR}{\sqrt{A^2 R^2 \omega^2 + 1}}$$

(S4.6)

Solving $(S4.6)$ for $A$ gives:

$$A = \frac{1}{R \omega b} \sqrt{(aR)^2 - b^2}$$

(S4.7)