CENG411: HOMEWORK 2

Please write down at the top of the page how many hours you spent on this homework.

1. (10 points)
An automobile has a weight of 3000 lb and is travelling at a velocity of 30 miles/hour. What constant braking force is required to bring it to a stop in 190 ft, if \( g = 9.81 \text{ m/s}^2 \)? What is the change in the internal energy of the car after it has come to rest?

2. (20 points)
[You may find it helpful to read Example 2.4, p. 25 in your textbook before working this problem.]
A gas with volume 1000 cm\(^3\) and pressure \( 32 \times 10^6 \text{ dynes/cm}^2 \) (state A) is expanded to 8000 cm\(^3\) at a pressure of \( 10^6 \text{ dynes/cm}^2 \) (state B). The process by which this occurs is quasi-static and no heat is exchanged with the environment. The mean pressure \( \bar{p} \) of the gas is found to change with its volume \( V \) according to the relation

\[
\bar{p} = \alpha V^{-5/3}
\]  

(1)

where \( \alpha \) is a constant. Find the quasi-static work done and the net heat absorbed by the system in each of the following processes, all of which take the system from macrostate \( A \) to macrostate \( B \).

a) The system is expanded from its original to final volume, heat being added to maintain the pressure constant. The volume is then kept constant, and heat is extracted to reduce the pressure to \( 10^6 \text{ dynes/cm}^2 \).
b) The volume is increased and heat is supplied to cause the pressure to decrease linearly with the volume.
c) The two steps of process (a) are performed in the opposite order.

3. FYI (No credit given)
Consider the case of a gas of \( N \) identical molecules enclosed in a container of volume \( V \). The energy of the system can be written as
\[ E = K + U + E_{\text{int}} \]  

where \( K \) is the total kinetic energy of the molecules. If the momentum of the center of mass of the \( i \)th molecule is denoted by \( \mathbf{p}_i \), then \( K \) depends only on these momenta and is given by

\[ K = K(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N) = \frac{1}{2m} \sum_{i=1}^{N} \mathbf{p}_i^2 \]  

The quantity \( U = U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \) represents the potential energy of mutual interaction between molecules. It depends on the relative separations between molecules i.e., on the center-of-mass positions \( \mathbf{r}_i \). If the molecules are not monatomic, the atoms of each molecule can also rotate and vibrate relative to its center of mass, and \( E_{\text{int}} \) represents the energy of the molecules due to such intramolecular motion.

Consider a monatomic gas \( (E_{\text{int}} = 0) \), where the mutual energy of interaction between the molecules is negligible \( (U \approx 0) \). The molecules are then said to form an “ideal” gas. This situation can be achieved physically in the limit where the concentration \( N/V \) of the molecules is made sufficiently small, for then the mean separation between molecules becomes so large that their mutual interaction is negligibly small.

**What is the number of states (at a given energy) \( \Omega(E) \) for such an ideal gas?** \( \Omega(E) \) is defined as the number of states whose energy lies between \( E \) and \( E + \delta E \), \( \delta E \) being the precision within which one chooses to measure the energy of the system. To calculate this, realise that (within the classical mechanics approximation) \( \Omega(E) \) is proportional to the volume of “phase space” i.e. two-dimensional space in \( \mathbf{r} \) and \( \mathbf{p} \):

\[ \Omega(E) = \int_{E}^{E+\delta E} \ldots \int \; d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_N d^3\mathbf{p}_1 \ldots d^3\mathbf{p}_N \]  

Here we have expressed the three-dimensional volume elements in terms of the three cartesian components of the position and momentum vectors using the abbreviation

\[ d^3\mathbf{r}_i = dx_idy_jdz_k \]  
\[ d^3\mathbf{p}_i = dp_{ix}dp_{iy}dp_{iz} \]
The integrand extends over all co-ordinates and momenta which are such that the total energy lies in between \( E \) and \( E + \delta E \).

a) Is the total energy dependent on the positions \( \mathbf{r}_i \) of the molecules?

b) To do the integrals over the position co-ordinates, relate each integral over \( \mathbf{r}_i \) to the container volume \( V \). How many such integrals are there? (We assume that all the molecules are confined within the container). Therefore how does \( \Omega(E) \) depend on the volume?

c) Is your result from (b) physically reasonable? For example, if the kinetic energy of each molecule is held fixed, and the volume of the box is doubled, how many more states become accessible to each molecule?

d) To do the integrals in momentum space, realise that for \( U = E_{\text{int}} = 0 \), the total energy \( E \) is just the sum of the kinetic energy of each particle so that

\[
2mE = \sum_{i=1}^{N} \sum_{\alpha=1}^{3} p_{i\alpha}^2
\]

since \( \mathbf{p}_i = p_{ix}^2 + p_{iy}^2 + p_{iz}^2 \). Let \( f \) denote the number of square terms given by the sum in equation 7.

For \( E = \text{constant} \), equation 7 then describes in the \( f \)-dimensional space of the momentum components, a sphere of radius \( R(E) = (2mE)^{1/2} \). The integral over the momenta is then proportional to the volume of a spherical shell lying between a sphere of radius \( R(E) \) and \( R(E + \delta E) \). (One way to see this is to imagine a single particle of mass \( m \) and energy \( E \), moving in two dimensions. Here \( (1/2m)(p_{x}^2 + p_{y}^2) = E \), so that a circle of radius \( (2mE)^{1/2} \) is traced out in \( p_x - p_y \) space.)

Hint: The volume of a sphere of radius \( R \) in three dimensions is proportional to \( R^3 \). What is the volume of a sphere in \( f \) dimensions proportional to?

e) You should thus obtain an expression for \( \Omega(E) \) in terms of \( V, N \) and \( E \) which involves an unknown proportionality constant independent of \( V \) and \( E \). Does \( \Omega(E) \) vary rapidly or slowly with \( E \)? Why?

4. Summary (3 points).

Summarize in two or three lines what you learnt by solving problems 1 and 2.

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