EXERCISE 2: Answers

1. Determine the order of magnitude of each of the following sequences of real numbers, as $t \to \infty$:

   (a) $y_t = t$
   The order of a sequence of reals is given by the power of $t$ that yields a limit when divided into the sequence. For this sequence $y_t/t = 1$ has a limit of 1 so $y_t = O(t)$.

   (b) $y_t = 3 + t^2$. Here, $\lim_{n\to\infty} y_t/t^2 = 1$, so $y_t = O(t^2)$.

   (c) $y_t = 3 + 1/t^2$. Here, $\lim_{n\to\infty} y_t = 3$, so $y_t = O(1)$.

   (d) $y_t = t + t^{1/2}$. Here, $\lim_{n\to\infty} y_t/t = 1$, so $y_t = O(t)$.

   (e) $y_t = (1 + 2t + t^2)/(1 + 2t^2)$. Since the $t^2$ terms dominate in the numerator and denominator, then $\lim_{n\to\infty} y_t = 1/2$, $y_t = O(1)$.

2. Suppose $x_t \sim i.i.d$, $E[x_t] = \mu$, and $V[x_t] = \sigma^2$ then determine the order in probability of the following sequences of random variables:

   (a) $x_1 - \mu$
   The order in probability of a sequence of random variables is the power of $n$ that yields a sequence with bounded or stable limiting distribution. Here, since $x_1$ and hence $x_1 - \mu$ have a stable distribution already $x_1 - \mu = O_p(1)$.

   (b) $\bar{X}_n = \sum_{t=1}^n x_t/n$. Rewriting $\bar{X}_n = \mu + (\bar{X}_n - \mu) = O(1) + O_p(1/\sqrt{n}) = O_p(1)$, where we use the result from (c) below.

   (c) $\bar{X}_n - \mu$. Here since $x_t$ is i.i.d., and has first two moments then $(\bar{X}_n - \mu)/(1/\sqrt{n}) = n^{1/2}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ then $\bar{X}_n - \mu = O_p(1/\sqrt{n})$.

   (d) $(\bar{X}_n)^2 - \mu^2$. Adding and subtracting we get $((\bar{X}_n - \mu) + \mu)^2 - \mu^2 = (\bar{X}_n - \mu)^2 + 2\mu(\bar{X}_n - \mu) + \mu^2 - \mu^2 = (\bar{X}_n - \mu)^2 + 2\mu(\bar{X}_n - \mu)$. Now the second term is $O_p(1/\sqrt{n})$ from (c) and the first must therefore be $O_p(1/n)$, so the second dominates and $(\bar{X}_n)^2 - \mu^2 = O_p(1/\sqrt{n})$.

   (e) $\sum_{t=1}^n (x_t - \mu)^2/n - \sigma^2$. Define $w_t = (x_t - \mu)^2$ as an i.i.d. random variable with expectation $\sigma^2$ and apply Khintchine’s theorem to find $\lim_{n \to \infty} \overline{w}_n = \sigma^2$ which means $\overline{w}_n - \sigma^2 = o_p(1)$. If we also assume a variance for $w_t$ (fourth moment of $x_t$) and apply the results from (c) to this new variable then $\sum_{t=1}^n (x_t - \mu)^2/n - \sigma^2 = \overline{w}_n - \sigma^2 = O_p(1/\sqrt{n})$.

3. Suppose $x_t \sim N(\mu, \sigma^2) \sim i.i.d.$
(a) Show that $\bar{X} = x_1 + 1/n$ is an asymptotically unbiased estimator of $\mu$ but is not consistent. Now $E[x_1] = \mu$ so $E[\bar{X}] = \mu + 1/n$ so $\lim_{n\to\infty} E[\bar{X}] = \mu$ and the estimator is asymptotically unbiased. But $\bar{X} \sim N(\mu + 1/n, \sigma^2)$ $\xrightarrow{d} N(\mu, \sigma^2)$ so the estimator’s distribution is not collapsing around the target and is therefore not consistent.

(b) Determine the limiting distribution of $[n^{1/2}(\bar{X}_n - \mu)]$. Since the $x_t$ are jointly normal any linear combination of them, including the average will also be normal also. Thus $\bar{X}_n \sim N(\mu, \sigma^2/n)$ and $n^{1/2}(\bar{X}_n - \mu) \sim N(0, \sigma^2)$ $\xrightarrow{d} N(0, \sigma^2)$.

(c) Show that $\bar{X}_n$ is asymptotically efficient relative to $\bar{X}$. Both $\bar{X}$ and $\bar{X}_n$ are asymptotically unbiased and normal thus we choose the one with the smaller variance $\bar{X}_n$.

(d) Determine the limiting distribution of $[n^{1/2}(\bar{X}_n + \bar{X}/n - \mu)]$. Rewrite $n^{1/2}(\bar{X}_n + \bar{X}/n - \mu) = n^{1/2}(\bar{X}_n - \mu) + \bar{X}/n$ and note that the first is $O_p(1)$ while the second in $O_p(1/\sqrt{n})$. Thus $[n^{1/2}(\bar{X}_n + \bar{X}/n - \mu)]$ $\xrightarrow{d} N(0, \sigma^2)$, which is the same as the limiting distribution of the first term.

(e) Determine the limiting distribution of $[(\bar{X}_n - \mu)/(s^2/n)]$ if $\text{plims}^2 = \sigma^2$. Rewrite $n_{\rightarrow\infty} (\bar{X}_n - \mu)/(s^2/n)^{1/2} = (\bar{X}_n - \mu)/(s^2/n)^{1/2} \cdot (\sigma^2/n)^{1/2}/(s^2/n)^{1/2}$. The first component $(\bar{X}_n - \mu)/(s^2/n)^{1/2}$ $\xrightarrow{d} N(0, 1)$ while the second $(\sigma^2/n)^{1/2}/(s^2/n)^{1/2}$ $\xrightarrow{p} 1$, so $(\bar{X}_n - \mu)/(s^2/n)^{1/2}$ $\xrightarrow{d} N(0, 1)$.

4. Suppose $x_t \sim i.i.d$ with $E[x_t] = \mu$ and $V[x_t] = \sigma^2$ but not necessarily normal

(a) Show that $\text{plim}_{n \to \infty} \bar{X}_n = \mu$. Since $x_t \sim i.i.d$ with $E[x_t] = \mu$ then the sample mean must be consistent for the population mean by Khintchine’s theorem. We can also use convergence in quadratic mean since a variance exists.

(b) Prove that $s^2 = \sum_{t=1}^{n}(x_t - \bar{X}_n)^2/(n-1)$ is consistent for $\sigma^2$. Rewrite $s^2 = \sum_{t=1}^{n}(x_t - \bar{X}_n)^2/(n-1) = \frac{n}{n-1} \left[ \frac{1}{n} \sum_{t=1}^{n}(x_t - \mu)^2 + \frac{n}{n-1} \frac{\sum_{t=1}^{n}(x_t - \mu)(\mu - \bar{X}_n)(\mu - \bar{X}_n)\right]$. The ratio $\frac{n}{n-1} \to 1$, the first term in brackets $\frac{1}{n} \sum_{t=1}^{n}(x_t - \mu)^2$ $\xrightarrow{p} \sigma^2$ by LLN and the last is $O_p(1/n)$ as is the second. Thus $s^2 = \sum_{t=1}^{n}(x_t - \bar{X}_n)^2/(n-1) \xrightarrow{p} \sigma^2$.

(c) Determine the limiting distribution of $[n^{1/2}(\bar{X}_n - \mu_0)]$ under both $H_0 : \mu = \mu_0$ and $H_1 : \mu = \mu_0 + \gamma/\sqrt{n}$. Rewrite $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) + n^{1/2}(\mu - \mu_0)$. Under $H_0$ the second term is zero and $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$. Under $H_1$ the second term becomes $n^{1/2}(\mu - \mu_0) = n^{1/2}(\gamma/\sqrt{n}) = \gamma$ so $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) + \gamma \xrightarrow{d} N(\gamma, \sigma^2)$. We have a shifted normal asymptotically under the alternative.

(d) Determine the limiting distribution of $[(\bar{X}_n - \mu)/(s^2/n)^{1/2}]$. This the same as (3.e).