Econ 409  
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Spring 2013

EXERCISE 3: Answers

1. Determine the order of magnitude of each of the following sequences of real numbers, as \( t \to \infty \):
   a. \( y_t = t \)
      
      The order of a sequence of reals is given by the power of \( t \) that yields a limit when divided into the sequence. For this sequence \( y_t/t = 1 \) has a limit of \( 1 \) so \( y_t = O(t) \).
   b. \( y_t = 3 + t^2 \). Here, \( \lim_{n \to \infty} y_t/t^2 = 1 \), so \( y_t = O(t^2) \).
   c. \( y_t = 3 + 1/t^2 \). Here, \( \lim_{n \to \infty} y_t = 3 \), so \( y_t = O(1) \).
   d. \( y_t = t + t^{1/2} \). Here, \( \lim_{n \to \infty} y_t/t = 1 \), so \( y_t = O(t) \).
   e. \( y_t = (1 + 2t + t^2)/(1 + 2t^2) \). Since the \( t^2 \) terms dominate in the numerator and denominator, then \( \lim_{n \to \infty} y_t = 1/2 \), \( y_t = O(1) \).

2. Suppose \( x_t \sim i.i.d., E[x_t] = \mu \), and \( V[x_t] = \sigma^2 \) then determine the order in probability of the following sequences of random variables:
   a. \( x_1 - \mu \)
      
      The order in probability of a sequence of random variables is the power of \( n \) that yields a sequence with bounded or stable limiting distribution. Here, since \( x_1 \) and hence \( x_1 - \mu \) have a stable distribution already \( x_1 - \mu = O_p(1) \).
   b. \( \bar{x}_n = \sum_{t=1}^n x_t/n \). Rewriting \( \bar{x}_n = \mu + (\bar{x}_n - \mu) = O(1) + O_p(1/\sqrt{n}) = O_p(1) \), where we use the result from (c) below.
   c. \( \bar{x}_n - \mu \). Here since \( x_t \) is i.i.d., and has first two moments then \( (\bar{x}_n - \mu)/(1/\sqrt{n}) = n^{1/2}(\bar{x}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \) then \( \bar{x}_n - \mu = O_p(1/\sqrt{n}) \).
   d. \( (\bar{x}_n)^2 - \mu^2 \). Adding and subtracting we get \( (\bar{x}_n - \mu + \mu)^2 - \mu^2 = (\bar{x}_n - \mu)^2 + 2\mu(\bar{x}_n - \mu) + \mu^2 - \mu^2 = (\bar{x}_n - \mu)^2 + 2\mu(\bar{x}_n - \mu) \). Now the second term is \( O_p(1/\sqrt{n}) \) from (c) and the first must therefore be \( O_p(1/n) \), so the second dominates and \( (\bar{x}_n)^2 - \mu^2 = O_p(1/\sqrt{n}) \).
   e. \( \sum_{t=1}^n (x_t - \mu)^2/n - \sigma^2 \). Define \( w_t = (x_t - \mu)^2 \) as an i.i.d. random variable with expectation \( \sigma^2 \) and apply Khintchine’s theorem to find \( \text{plim}_{n \to \infty} \bar{w}_n = \sigma^2 \) which means \( \bar{w}_n - \sigma^2 = o_p(1) \). If we also assume a variance for \( w_t \) (fourth moment of \( x_t \)) and apply the results from (c) to this new variable then \( \sum_{t=1}^n (x_t - \mu)^2/n - \sigma^2 = \bar{w}_n - \sigma^2 = O_p(1/\sqrt{n}) \).

3. Suppose \( x_t \sim N(\mu, \sigma^2) \sim i.i.d. \)
   a. Show that \( \bar{X} = x_1 + 1/n \) is an asymptotically unbiased estimator of \( \mu \) but is not consistent.
      
      Now \( E[x_1] = \mu \) so \( E[\bar{X}] = \mu + 1/n \) so \( \lim_{n \to \infty} E[\bar{X}] = \mu \) and the estimator is asymptotically unbiased. But \( \bar{X} \sim N(\mu + 1/n, \sigma^2) \xrightarrow{d} N(\mu, \sigma^2) \) so the estimator’s distribution is not collapsing around the target and is therefore not consistent.
   b. Determine the limiting distribution of \( [n^{1/2}(\bar{x}_n - \mu)] \). Since the \( x_t \) are jointly normal any linear combination of them, including the average will also be normal also. Thus \( \bar{x}_n \sim N(\mu, \sigma^2/n) \) and \( n^{1/2}(\bar{x}_n - \mu) \sim N(0, \sigma^2) \xrightarrow{d} N(0, \sigma^2) \).
   c. Show that \( \bar{X}_n \) is asymptotically efficient relative to \( \bar{X} \). Both \( \bar{X} \) and \( \bar{X}_n \) are asymptotically unbiased and normal thus we choose the one with the smaller variance \( \bar{X}_n \).
   d. Determine the limiting distribution of \( [n^{1/2}(\bar{x}_n + \bar{X}/n - \mu)] \). Rewrite \( n^{1/2}(\bar{x}_n + \bar{X}/n - \mu) \).
4. Suppose $x_t \sim i.i.d$ with $E[x_t] = \mu$ and $V[x_t] = \sigma^2$ but not necessarily normal

a. Show that $\text{plim} \bar{X}_n = \mu$. Since $x_t \sim i.i.d$ with $E[x_t] = \mu$ then the sample mean must be consistent for the population mean by Khintchine’s theorem. We can also use convergence in quadratic mean since a variance exists.

b. Prove that $s^2 = \sum_{t=1}^{n} (x_t - \bar{X}_n)^2 / (n - 1)$ is consistent for $\sigma^2$. Rewrite $s^2 = \sum_{t=1}^{n} (x_t - \bar{X}_n)^2 / (n - 1) = \frac{n}{n-1} \left[ \frac{1}{n} \sum_{t=1}^{n} (x_t - \mu)^2 + \frac{2}{n} \sum_{t=1}^{n} (x_t - \mu)(\mu - \bar{X}_n) \right]$. The ratio $\frac{s^2}{n-1} \to 1$, the first term in brackets $\frac{1}{n} \sum_{t=1}^{n} (x_t - \mu)^2 \xrightarrow{p} \sigma^2$ by LLN and the last is $O_p(1/n)$ as is the second. Thus $s^2 = \sum_{t=1}^{n} (x_t - \bar{X}_n)^2 / (n - 1) \xrightarrow{p} \sigma^2$.

c. Determine the limiting distribution of $[n^{1/2}(\bar{X}_n - \mu_0)]$ under both $H_0 : \mu = \mu_0$ and $H_1 : \mu = \mu_1 = \mu_0 + \gamma / \sqrt{n}$. Rewrite $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) + n^{1/2}(\mu - \mu_0)$. Under $H_0$ the second term is zero and $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$. Under $H_1$ the second term becomes $n^{1/2}(\mu - \mu_0) = n^{1/2}(\gamma / \sqrt{n}) = \gamma$ so $n^{1/2}(\bar{X}_n - \mu_0) = n^{1/2}(\bar{X}_n - \mu) + \gamma \xrightarrow{d} N(\gamma, \sigma^2)$. We have a shifted normal asymptotically under the alternative.

d. Determine the limiting distribution of $[(\bar{X}_n - \mu) / (s^2/n)^{1/2}]$. This the same as (3.e).