EXERCISE 1

This an exercise in the theory and application of the normal approximation when the underlying distribution is not normal. It also gives you an opportunity to refresh your memory on how to conduct a hypothesis test and construct a confidence interval.

1. Consider a Bernoulli random variable. Specifically \(X \in \{0, 1\}\) and \(Pr(x = 1) = \pi\) and \(Pr(x = 0) = 1 - \pi\).

   (a) Obtain the population mean and variance of the distribution.
   (b) Obtain the mean and variance of the sample mean \(\hat{\pi} = \frac{\sum X}{n}\) based on a sample size of \(n\).
   (c) Using the law of large numbers show that \(\lim_{n \to \infty} \hat{\pi} = \pi\).
   (d) Using the central limit theorem obtain the limiting distribution of \((\hat{\pi} - \pi)/(\pi(1 - \pi)/n)^{1/2}\).
   (e) What should be the limiting distribution of \((\hat{\pi} - \pi)/(\hat{\pi}(1 - \hat{\pi})/n)^{1/2}\).

2. In a survey of 400 likely voters, 215 responded that they would vote for the incumbent and 185 responded that they would vote for the challenger. Let \(\pi\) denote the fraction of all likely voters that preferred the incumbent at the time of the survey, and let \(\hat{\pi}\) be the fraction of the survey respondents that preferred the incumbent.

   (a) Use the survey results to obtain a value for the estimator \(\hat{\pi}\).
   (b) Based on your findings in (1), obtain an appropriate estimate of the variance of \(\hat{\pi}\).
   (c) For \(\alpha = .05\), test the null \(H_0 : \pi = 0.5\) vs. \(H_1 : \pi \neq 0.5\).
   (d) For \(\alpha = .05\), test the null \(H_0 : \pi = 0.5\) vs. \(H_1 : \pi > 0.5\).
   (e) Do the results from (2.c) and (2.d) agree? Why or why not?

3. Using the data in (2.):

   (a) Construct a 95\% confidence interval for \(\pi\).
   (b) What does the interval tell us?
   (c) Construct a 99\% confidence interval for \(\pi\).
   (d) Why is the interval in (3.b) wider?
   (e) Using only the calculation from this problem, test the hypothesis \(H_0 : \pi = 0.5\) vs. \(H_1 : \pi \neq 0.5\) at the 5\% significance level.