Genericity of Critical Types

Yi-Chun Chen Alfredo Di Tillio Eduardo Faingold
Siyang Xiong

September 2008

Abstract

Ely and Peski (2008) offers an insightful characterization of critical types: a type is critical if and only if it has common-p belief on some closed, proper subset of the universal type space. Thus, all types ever considered in applications are critical. On the other hand, Ely and Peski (2008) show that the critical types form a meager subset of the universal type space under the product topology. We propose two ways to resolve this puzzle by proving two genericity results for critical types. First, under the strategic topology due to (Dekel, Fudenberg, and Morris, 2006, *Theoretical Economics* **1** (2006) 275–309), the set of critical types is open and dense in the universal type space. Second, geometrically, the set of critical types is prevalent (i.e. the complement of a finitely shy set).

1 Introduction

Ely and Peski (2008) offer a very clean characterization of critical types: a type is critical if and only if it has common-p belief on some proper closed subset of the universal type space. Consequently, all type spaces ever considered in applications consists entirely of critical types. However, they also offer a puzzling result that critical types are non-generic, i.e., they form a meager set in the universal type space under the product topology. We propose two ways to resolve this puzzle by proving two genericity results for critical types. First, under the stronger strategic topology due to (Dekel, Fudenberg, and Morris, 2006, *Theoretical Economics* **1** (2006) 275–309), critical types is open and dense in the universal type space. Second, if we identify the set of types with $\Delta(\Omega \times U_{-i}(\Omega))$ under the standard Mertens-Zamir homeomorphism, the set of regular types (i.e., the complement of critical types) form a proper face in $\Delta(\Omega \times U_{-i}(\Omega))$ and hence it is finitely shy.

2 Preliminaries

Let d denote the product metric on the universal type space.

For any set $W \subset U_i(\Omega)$, denote the ε-open ball containing W under the product topology by W^ε, i.e.,

$$W^\varepsilon := \{t_i \in U_i(\Omega) : d(t_i, t_i') < \varepsilon \text{ for some } t_i' \in W\}.$$

Definition 1 (Common belief convergence) *A sequence of types $(u_i^n)_{n=1}^\infty$ converges to a type u_i in common-p belief if for any $p' > 0$ and any closed proper subset $W \subset U_i(\Omega)$ with $u_i \in C_i^{p'}(W)$ and any $\varepsilon \in (0, p')$, there exists a positive integer N such that $u_i^n \in C_i^{p'-\varepsilon}(W^\varepsilon)$ for any $n \geq N$.*

3 The First Genericity Result

In this section, we prove critical types are generic in a topological sense.

Theorem 1 *Under the strategic topology, critical types contain an open and dense subset of the universal type space.*

By Theorem 4 of Dekel, Fudenberg, and Morris (2006), finite types are dense in the universal type space under the strategic topology. Since finite types are critical by Theorem 3 of Ely and Peski (2008), critical types are dense under the strategic topology. Therefore, Theorem 1 is a direct consequence of the following Proposition.

Proposition 1 *The strategic closure of regular types contains no finite types.*

To prove Proposition 1, we need the following Proposition, whose proof is relegated to Appendix 1.

Proposition 2 $(u_i^n)_{n=1}^{\infty}$ *converges to* u_i *under the strategic topology only if* $(u_i^n)_{n=1}^{\infty}$ *converges to* u_i *in common-*p *belief.*

Proof of Proposition 1. Suppose instead that a sequence of regular types $(u_i^n)_{n=1}^{\infty}$ converges to a finite type u_i under the strategic topology. Since u_i is a finite type, $u_i \in C_i^p(W)$ for a finite set $W \subset U_i(\Omega)$. Moreover, since $(u_i^n)_{n=1}^{\infty}$ converges to u_i under the strategic topology, $(u_i^n)_{n=1}^{\infty}$ also converges to u_i in common-p belief by Proposition 2. Since W is finite, for sufficiently small $\varepsilon > 0$, the product closure of W^{ε}, denoted by $\overline{W^{\varepsilon}}$, is still a proper closed subset of $U_i(\Omega)$, and moreover, $p - \varepsilon > 0$. Since $(u_i^n)_{n=1}^{\infty}$ converging to u_i in common-p belief, by Definition 1, there exists a positive integer N such that $u_i^n \in C_i^{p-\varepsilon}\left(\overline{W^{\varepsilon}}\right)$. Hence, u_i^n is a critical type for all $n \geq N$ by Ely and Peski (2008), which contradicts to the assumption that u_i^n is regular for all n.■

Remark. Our proof is still valid if we consider the solution concept of IIR instead of ICR.

4 The Second Genericity Result

In the section, we prove that the set of critical types, viewed as a subset of $\Delta(\Omega \times U_{-i}(\Omega))$ under the standard Mertens-Zamir homeomorphism, is generic in a geometric sense, i.e., it is *prevalent*. Throughout this section, for any two types $u_i, u_i' \in U_i(\Omega)$ and $\alpha \in (0,1)$, define

$$\alpha u_i + (1-\alpha) u_i' \equiv \pi_i^{-1} [\alpha \pi_i(u_i) + (1-\alpha) \pi_i(u_i')]$$

where π is the Mertens-Zamir homeomorphism between $U_i(\Omega)$ and $\Delta(\Omega \times U_{-i}(\Omega))$. Hence, $\alpha u_i + (1-\alpha) u_i' \in U_i(\Omega)$. The following proposition shows that critical types form a proper face in the universal types space.

Proposition 3 *For any $\alpha \in (0,1)$, $u_i'' = \alpha u_i + (1-\alpha) u_i'$ is regular iff both u_i and u_i' are regular.*

Proof. We prove first the "only if" part. Suppose u_i is critical. By Theorem 3 of Ely and Peski (2008), $u_i \in C_i^p(W_i)$ for some closed proper subset $W_i \subset U_i(\Omega)$. By Lemma 6 of Ely and Peski (2008), $C_i^p(W_i) = W_i \cap B_i^p C_{-i}^p(W_i \times U_{-i}(\Omega))$. Hence,

$$u_i'' \in (W_i \cup \{u_i''\}) \cap B_i^{p'} C_{-i}^{p'}((W_i \cup \{u_i''\}) \times U_{-i}(\Omega))$$

where $p' = \min\{p, \alpha\}$. By Lemma 6 of Ely and Peski (2008), $u_i'' \in C_i^{p'}(W_i \cup \{u_i''\})$. Since $W_i \cup \{u_i''\}$ is a proper closed subset of $U_i(\Omega)$, u_i'' is critical by Theorem 3 of Ely and Peski (2008).

We then turn to show the "if" part. Suppose that u_i'' is critcal. Then, By Theorem 3 of Ely and Peski (2008), $u_i \in C_i^p(W_i)$ for some closed proper subset $W_i \subset U_i(\Omega)$. By Lemma 6 of Ely and Peski (2008), $C_i^p(W_i) = W_i \cap B_i^p C_{-i}^p(W_i \times U_{-i}(\Omega))$. Since $u_i'' = \alpha u_i + (1-\alpha) u_i'$, either $u_i \in B_i^p C_{-i}^p(W_i \times U_{-i}(\Omega))$ or $u_i' \in B_i^p C_{-i}^p(W_i \times U_{-i}(\Omega))$. Hence, by Lemma 6 of Ely and Peski (2008), either u_i or u_i' is critical.■

Theorem 2 *The set of critcal types is prevalent.*

Proof. This is a direct consequence of our Proposition 3 and Heifetz and Neeman (2006)'s Lemma 1.■

A Appendix

A.1 Proof of Proposition 2

Proof. Suppose that $(u_i^n)_{n=1}^{\infty}$ does not converge to u_i in common-p belief. Then, there exists a $W \subset U_i(\Omega)$ with $u_i \in C_i^p(W)$ and some $\varepsilon > 0$ such that there is a subsequence, say itself, such that $u_i^n \notin C_i^{p-\varepsilon}(W^\varepsilon)$ for all n. We will show that $(u_i^n)_{n=1}^{\infty}$ does not converge to u_i under strategic topology. Since W^ε is open, $U_i(\Omega) \backslash W^\varepsilon$ is closed and obviously also nonempty. Since W and $U_i(\Omega) \backslash W^\varepsilon$ are two disjoint nonempty closed subset of $U_i(\Omega)$, by Lemma 3, there exist $\gamma > 0$ and a game $G = \left\langle (A_i)_{i=1,2}, (g_i)_{i=1,2} \right\rangle$ such that properties 1–3 in Lemma 3 are satisfied. Define G^* in the same way as defined in the proof of Lemma 2 of Ely and Peski (2008). Define the following set of hierarchies

$$\begin{aligned} V &= \left\{ u_i : A^{0*} \times A^1(u_i) \subseteq R_i(u_i|G, 0) \right\}; \\ V(\gamma) &= \left\{ u_i : A^{0*} \times A^1(u_i) \cap R_i(u_i|G, \gamma) = \varnothing \right\}. \end{aligned}$$

By the property 3 of Lemma 3, $W \subseteq V \subseteq V(\gamma) \subseteq W^\varepsilon$. Let $Z = A^{0*} \times A^1 \times \{1\}$. By the proof of Lemma 2 of Ely and Peski (2008),

1. If $u_i' \in C_i^p(V)$, then $Z \cap R_i(u_i'|G^*, 0) \neq \varnothing$;
2. If $u_i' \notin C_i^{p-2\gamma(1-p)}(V(\gamma))$, then $Z \cap R_i(u_i'|G^*, \gamma) = \varnothing$.

Note that γ can be chosen to be small so that $2\gamma(1-p) < \varepsilon$ without losing property 3 of Lemma 3. Since $u_i^n \notin C_i^{p-\varepsilon}(W^\varepsilon)$ for all n and $V(\gamma) \subseteq W^\varepsilon$, $u_i^n \notin C_i^{p-2\gamma(1-p)}(V(\gamma))$. Therefore, $(u_i^n)_{n=1}^{\infty}$ does not converge to u_i under strategic topology.$\blacksquare$

A.2 Proof of Lemma 3

Lemma 1 *Suppose that $v_i \neq u_i$. There are open neighborhoods $V_i \ni v_i$ and $U_i \ni u_i$, game $\overline{G} = \left(\overline{A}_j, \overline{g}_j\right)$, action $a_i \in \overline{A}_i$, $\varepsilon > 0$ and m such that*

$$a_i \in R_i^m\left(u_i'|\overline{G}, 0\right) \text{ for all } u_i' \in U_i \text{ and } a_i \notin R_i^m\left(v_i'|\overline{G}, \varepsilon\right) \text{ for all } v_i' \in V_i\,.$$

Proof. Since $v_i \not\leq u$, by Lemma 17 of Ely and Peski (2008), there is a game $\overline{G}'$, an action a_i', $\varepsilon > 0$, and some positive integer m such that $a_i' \in R_i^m\left(u_i|\overline{G}', 0\right)$ and $a_i' \notin R_i^m\left(v_i|\overline{G}', \varepsilon\right)$. By Lemma 16 of Ely and Peski (2008), there is some open neighborhoods $V_i \ni v_i$ and $U_i \ni u_i$ such that $a_i \in R_i^m\left(u_i'|\overline{G}, \varepsilon/3\right)$ for all $u_i' \in U_i$ and $a_i \notin R_i^m\left(v_i'|\overline{G}, 2\varepsilon/3\right)$ for all $v_i' \in V_i$. Then, by Lemma 4 of **?**, there is a game $\overline{G} = \left(\overline{A}_j, \overline{g}_j\right)$ and an action $a_i \in \overline{A}_i$ such that $a_i \in R_i^m\left(u_i'|\overline{G}, 0\right)$ for all $u_i' \in U_i$ and $a_i \notin R_i^m\left(v_i'|\overline{G}, \varepsilon/3\right)$ for all $v_i' \in V_i$.■

Lemma 2 *Suppose that $v_i \neq u_i$. There are $\varepsilon > 0$, open neighborhoods $V_i \ni v_i$ and $U_i \ni u_i$, a game $G = (A_j, g_j)$ such that $A_i = A_i^0 \times A_i^1$ and*

1. *For any $a_{-i} \in A_{-i}$, any $a_i^1 \in A_i^1$ and any $a_i^0, a_i^{0\prime} \in A_i^0$, any ω,*
$$g_{-i}\left(a_{-i}, \left(a_i^0, a_i^1\right), \omega\right) = g_{-i}\left(a_{-i}, \left(a_i^{0\prime}, a_i^1\right), \omega\right).$$
2. *There are correspondence $A^0 : U_i(\Omega) \rightrightarrows A_i^0$, $A^1 : U_i(\Omega) \rightrightarrows A_i^1$ such that for all u_i',*
$$R_i\left(u_i'|G, 0\right) = A^0\left(u_i'\right) \times A^1\left(u_i'\right).$$
3. *There is an action $a_i^{0*} \subseteq A_i^0$ such that*
$$\left\{a_i^{0*}\right\} \times A^1\left(u_i'\right) \subseteq R_i\left(u_i'|G, 0\right) \text{ for all } u_i' \in U_i;$$
$$\left\{a_i^{0*}\right\} \times A^1\left(v_i'\right) \cap R_i\left(v_i'|G, 0\right) = \varnothing \text{ for all } v_i' \in V_i.$$

Proof. This follows from the proof of Lemma 19 of **?**.■

Lemma 3 *Fix player i. Let W_i and W_i' be nonempty closed subsets of $U_i(\Omega)$ such that $W_i' \cap W_i = \varnothing$, there are $\varepsilon > 0$ and a game $G = (A_j, g_j)$ such that $A_i = A_i^0 \times A_i^1$ and*

1. *For any $a_{-i} \in A_{-i}$, any $a_i^1 \in A_i^1$ and any $a_i^0, a_i^{0\prime} \in A_i^0$, any ω,*
$$g_{-i}\left(a_{-i}, \left(a_i^0, a_i^1\right), \omega\right) = g_{-i}\left(a_{-i}, \left(a_i^{0\prime}, a_i^1\right), \omega\right).$$
2. *There are correspondence $A^0 : U_i(\Omega) \rightrightarrows A_i^0$, $A^1 : U_i(\Omega) \rightrightarrows A_i^1$ such that for all u_i,*
$$R_i\left(u_i|G, 0\right) = A^0\left(u_i\right) \times A^1\left(u_i\right).$$

3. There is a nonempty subset $A_i^{0*} \subseteq A_i^0$ such that

$$A_i^{0*} \times A^1(u_i) \subseteq R_i(u_i|G,0) \text{ for all } u_i \in W_i;$$
$$A_i^{0*} \times A^1(v_i) \cap R_i(v_i|G,\varepsilon) = \varnothing \text{ for all } v_i \in W_i'$$

Proof. First, fix $u_i \in W_i$. Since W_i is an closed set and $W_i \cap W_i' = \varnothing$, by Lemma 2, for every $v_i \in W_i'$, there are $\varepsilon > 0$, open neighborhoods $V_i^{v_i} \ni v_i$ and $U_i^{u_i} \ni u_i$, a game $G^{v_i} = \left(A_j^{v_i}, g_j^{v_i}\right)$ such that $A_i^{v_i} = A_i^{v_i 0} \times A_i^{v_i 1}$ and an action $a_i^{v_i 0*}$ properties 1–3 in Lemma 2 are satisfied. Since $(V_i^{v_i})_{v_i \in W_i'}$ is an open cover of W_i', there is a finite subcover $V_i^{v_i^1}, ...V_i^{v_i^K}$ of W_i'. Let $U_i^1, ..., U_i^K$ be the corresponding open neighborhoods of u_i, $G^{v_i^1}, ..., G^{v_i^K}$ be the corresponding games (for u_i and v_i^k, $k = 1, ..., K$, respectively), and $a_i^{v_i^1 0*}, ..., a_i^{v_i^K 0*}$ be the corresponding actions in property 3 of Lemma 2. Let $G^{(u_i)}$ be the product game of $G^{v_i^1}, ..., G^{v_i^K}$. Define $A_i^{u_i 0*} = \left\{\left(a_i^{v_i^1 0*}, ..., a_i^{v_i^K 0*}\right)\right\}$. Let $U_i^{(u_i)} \equiv \cap_{k=1}^K U_i^k$. Then, properties 1 and 2 are satisfied for $G^{(u_i)}$ and

$$A_i^{0*} \times A^1(u_i') \subseteq R_i\left(u_i'|G^{(u_i)}, 0\right) \text{ for all } u_i' \in U_i^{(u_i)};$$
$$A_i^{u_i 0*} \times A^1(v_i) \cap R_i(v_i|G,0) = \varnothing \text{ for all } v_i \in W_i'.$$

Since $\left(U_i^{(u_i)}\right)_{u_i \in W_i}$ is an open cover of W_i, there is a finite subcover $U_i^{\left(u_i^1\right)}, ..., U_i^{\left(u_i^{K'}\right)}$ of W_i. Let G be the product game of $G^{\left(u_i^1\right)}, ..., G^{\left(u_i^{K'}\right)}$. Let

$$A_i^{0*} \equiv \left\{a_i^0 \in A_i^0 : a_i^{0k} \in A_i^{u_i^k 0*},\ a_i^0 \in A^0(u_i) \text{ for some } u_i\right\}.$$

We can verify that A_i^{0*} has the desired properties.■

A.3 An Alternative Proof of Proposition 2

We prove the contra-positive statement of Proposition 2: if $(t_n)_{n=1}^\infty$ does not *converge to* t in common-p belief, then $(t_n)_{n=1}^\infty$ does not *converge to* t under the strategic topology.

Suppose $(t_n)_{n=1}^\infty$ does not *converge to* t in common-p belief, i.e., there exists a product-closed $E \subsetneqq \mathcal{T}_i$ with $t \in C^p(E)$ and some $\varepsilon > 0$, but there is a subsequence $(t_{n_k})_{k=1}^\infty$ such that $t_{n_k} \notin C^p(E^\varepsilon)$ for all k. For notational ease, we use t_k to denote t_{n_k}.

We will show that $(t_k)_{k=1}^{\infty}$ does not converge to t under strategic topology, which proves that $(t_n)_{n=1}^{\infty}$ does not *converge to* t under the strategic topology.

To prove Proposition 2, we need the following proposition[1], whose proof is relegated to Appendix A.4.

Proposition 4 *For any two disjoint non-empty closed sets $U, V \subset \mathcal{T}_i$, there exist $\gamma > 0$ and a game $\widehat{G} = \left\langle \left(\widehat{A_i}\right)_{i=1,2}, (\widehat{g}_i)_{i=1,2} \right\rangle$ such that if an action $\widehat{a_i} \in \widehat{A_i}$ is $0-$rationalizable for some $t_i \in U$, then $\widehat{a_i}$ is not $\gamma-$rationalizable for $t_i' \in V$.*

Since E^{ε} is open, its complement denoted by $(E^{\varepsilon})^C$ is closed. Then E and $(E^{\varepsilon})^C$ are two disjoint non-empty closed sets. By Lemma **??**, there exist $\gamma > 0$ and a game $\widehat{G} = \left\langle \left(\widehat{A_i}\right)_{i=1,2}, (\widehat{g}_i)_{i=1,2} \right\rangle$ such that if an action $\widehat{a_i} \in \widehat{A_i}$ is $0-$rationalizable for some $t_i \in E$, then $\widehat{a_i}$ is not $\gamma-$rationalizable for $t_i' \in (E^{\varepsilon})^C$. Define

$$\Lambda = \bigcup_{t_i \in E} R_i\left(t_i, \widehat{G}, 0\right);$$
$$E_{\gamma} := \left\{t_i : R_i\left(t_i, \widehat{G}, \gamma\right) \bigcap \Lambda \neq \varnothing\right\}.$$

Define a new game $G^* = \left\langle (A_i^*)_{i=1,2}, (g_i^*)_{i=1,2} \right\rangle$ such that $A_i^* = \widehat{A_i} \times \{0,1\}$ and for $a^* = [(\widehat{a_i}, z_i), (\widehat{a_{-i}}, z_{-i})]$, we have

$$g_i^*(a^*, \theta) = \widehat{g_i}(\widehat{a_i}, \widehat{a_{-i}}, \theta) + \begin{cases} 1, & \text{if } z_i = z_{-i} = 1; \\ \frac{-p}{1-p}, & \text{if } z_i = 1 \text{ and } z_{-i} = 0; \\ 0, & \text{if } z_i = 0. \end{cases}$$

$$g_{-i}^*(a^*, \theta) = \widehat{g_{-i}}(\widehat{a_i}, \widehat{a_{-i}}, \theta) + \begin{cases} 1, & \text{if } z_i = z_{-i} = 1 \text{ and } \widehat{a_i} \in \Lambda; \\ \frac{-p}{1-p}, & \text{if } z_{-i} = 1 \text{ and } (\widehat{a_i} \notin \Lambda \text{ or } z_{-i} = 0); \\ 0, & \text{if } z_{-i} = 0. \end{cases}$$

Just like the proof in Ely and Peski (2008), we can show that

[1]Proposition 4 is a stronger version of DFM's lemma 4, which says that for any two distinct types t and s, there exist $\gamma > 0$ and a game $\widehat{G} = \left\langle \left(\widehat{A_i}\right)_{i=1,2}, (\widehat{g}_i)_{i=1,2} \right\rangle$ such that if an action $\widehat{a_i} \in \widehat{A_i}$ is $0-$rationalizable for some t, then $\widehat{a_i}$ is not $\gamma-$rationalizable for s.

1. If $t_i \in C^p(E)$, then $\left[\widehat{A_i} \times \{1\}\right] \bigcap R_i(t_i, G^*, 0) \neq \varnothing$;

2. If $t_i \notin C^p(E_\gamma)$, then $\left[\widehat{A_i} \times \{1\}\right] \bigcap R_i(t_i, G^*, \gamma) = \varnothing$.

Note that $E_\gamma \subset E^\varepsilon$, because for any $\widehat{a}_i \in \widehat{A_i}$ being $0-$rationalizable for some $t_i \in E$ is not $\gamma-$rationalizable for $t_i' \in (E^\varepsilon)^C$.

Recall that $t \in C^p(E)$ and $t_k \notin C^p(E^\varepsilon)$ for all k. Hence, $t_k \notin C^p(E_\gamma)$ for all k. Thus,

1. $\left[\widehat{A_i} \times \{1\}\right] \bigcap R_i(t, G^*, 0) \neq \varnothing$;

2. $\left[\widehat{A_i} \times \{1\}\right] \bigcap R_i(t_k, G^*, \gamma) = \varnothing$ for all k.

Therefore, $(t_k)_{k=1}^\infty$ does not converge to t under strategic topology.

A.4 The proof of Proposition 4

Proposition 4 is an immediate consequence of the following two lemmas. For the ease of exposition, we relegate the proofs of these lemmas to Appendix A.4 and A.4.

For any $E \subset \mathcal{T}_i$, define $T_{k,E}$ as $T_{k,E} := \{t \in \mathcal{T}_i : \pi^k(t) = \pi^k(t') \text{ for some } t' \in E\}$.

Lemma 4 *For any two disjoint closed sets $U, V \subset \mathcal{T}_i$, there exists some positive integer k such that U^k and V^k are two disjoint closed sets in $\mathcal{T}_i^k$.*

Lemma 5 *For any positive integer k and two disjoint non-empty closed sets $W, Z \subsetneqq \mathcal{T}^k$, there exist $\gamma > 0$ and a game $\widehat{G} = \left\langle \left(\widehat{A_i}\right)_{i=1,2}, (\widehat{g}_i)_{i=1,2} \right\rangle$ such that if an action $\widehat{a}_i \in \widehat{A_i}$ is $0-$rationalizable for some t with $\pi^k(t) \in W$, then $\widehat{a}_i$ is not $\gamma-$rationalizable for any t' with $\pi^k(t') \in Z$.*

The proof of Lemma 4 First, clearly, if $U, V \subset \mathcal{T}_i$ are closed, the two sets $U^k, V^k \subset \mathcal{T}_i^k$ are close for any k.

Second, for any $E \subset \mathcal{T}_i$, define $T_{k,E}$ as $T_{k,E} := \{t \in \mathcal{T}_i : \pi^k(t) = \pi^k(t') \text{ for some } t' \in E\}$. We prove the following claim.

Claim 1 *For any close $E \subset \mathcal{T}_i$, $\cap_{k=1}^{\infty} T_{k,E} = E$.*

Proof. First, we show that $\cap_{k=1}^{\infty} T_{k,\{t\}} = \{t\}$. Apparently, $\cap_{k=1}^{\infty} T_{k,\{t\}} \supset \{t\}$. Also, $\cap_{k=1}^{\infty} T_{k,\{t\}} \subset \{t\}$. Suppose not, i.e. there is $t' \in \cap_{k=1}^{\infty} T_{k,\{t\}}$, but $t' \neq t$. Since $t' \neq t$, $\pi^k(t) \neq \pi^k(t')$ for some k, which contradicts to $t' \in T_{k,\{t\}}$. Therefore, $\cap_{k=1}^{\infty} T_{k,\{t\}} \subset \{t\}$.

Second, by coherence, $T_{1,E} \supset T_{2,E} \supset T_{3,E} \supset ... \supset T_{k,E} \supset ... \supset E$. Thus, $\cap_{k=1}^{\infty} T_{k,E} \supset E$.

Third, we show that $\cap_{k=1}^{\infty} T_{k,E} \subset E$. Pick any $t \in \cap_{k=1}^{\infty} T_{k,E}$. Note, $T_{k,\{t\}} \cap E$ is closed, compact and non-empty[2]. Thus, $\varnothing \neq \cap_{k=1}^{\infty} \left[T_{k,\{t\}} \cap E\right] \subset \cap_{k=1}^{\infty} T_{k,\{t\}} = \{t\}$. Therefore, $\cap_{k=1}^{\infty} \left[T_{k,\{t\}} \cap E\right] = \{t\}$ and $\{t\} = \cap_{k=1}^{\infty} \left[T_{k,\{t\}} \cap E\right] \subset E$.■

Second, we prove the following claim.

Claim 2 *For any two disjoint closed sets $U, V \subset \mathcal{T}_i$, there exists some positive integer k such that $T_{k,U}$ and $T_{k,V}$ are disjoint.*

Proof. suppose otherwise, i.e., $T_{k,U} \bigcap T_{k,V} \neq \varnothing$ for all k. Hence, $\bigcap_{k \in \mathbb{Z}_+} [T_{k,U} \bigcap T_{k,V}] \neq \varnothing$. Then,

$$\varnothing \neq \bigcap_{k \in \mathbb{Z}_+} \left[T_{k,U} \bigcap T_{k,V}\right] = \left[\bigcap_{k \in \mathbb{Z}_+} T_{k,U}\right] \bigcap \left[\bigcap_{k \in \mathbb{Z}_+} T_{k,V}\right] = U \bigcap V, \tag{1}$$

where the last equality follows from Claim 1 and the fact that U and V are closed. (1) contradicts the the assumption that U and V are disjoint.■

Third, note that $[T_{k,U}]^k = U^k$ and $[T_{k,V}]^k = V^k$. Therefore, Lemma 4 is true.

[2]Consider the product toplogy on T^*. The function $\gamma : T^* \to \pi^k(T^*)$ such that $\gamma(t) = \pi^k(t)$ is continuous. Hence, $T_{k,\{t\}}$ is closed. Also, it is compact because T^* is compact.

The proof of Lemma 5 Following DFM, we use quadratic scoring rule to elicit type's k-th order belief. In this scoring rule game $\widehat{G} = \left\langle \left(\widehat{A_i}\right)_{i=1,2}, (\widehat{g}_i)_{i=1,2} \right\rangle$, the action set is $\widehat{A_i} = B^1 \times ... \times B^k \subset T^1 \times ... \times T^k$, where $\mathcal{T}_i^k$ is the space of k-th order belief and B^k is a set of finite discrete grids in T^k. Note that $B^k \subset \triangle\left(B^{k-1}\right)$.

For any type t_i, let $\left[b^1(t_i), ..., b^k(t_i)\right]$ denote an arbitrary 0-rationalizable strategy for t_i, where $b^1(t_i), ..., b^k(t_i)$ correspond to the 1st, ..., the k-th order beliefs respectively.

Let $(t_i)^k$ denote t_i's *true* k-th belief. We say $b^k(t_i)$ is t_i's k-th *reporting belief.* We define t_i's k-th *induced belief*, $I^k(t_i)$ as follows.

$$I^k(t_i)[E] = t_i\left[\{t_{-i} : b^{k-1}(t_i) \in E\}\right] \text{ for } E \subset T^{k-1}.$$

With the quadratic scoring rule, the players' 0-rationalizable strategy is to truthfully report their beliefs subject measurement errors due to finite grids, i.e., they chooses $b^k(t_i)$ to be the grid which is mostly closed to $I^k(t_i)$.

To define $\widehat{G}$, we only need to define the grids of $\widehat{A_i}$. We will show that the grids can be approximately chosen so that Lemma 5 is true. We establish this by the following three steps.

Step 1 For any k and $\delta > 0$, we can choose the grids so that $\rho^k\left[(t_i)^k, b^k(t_i)\right] < \delta$ and $\rho^k\left[(t_i)^k, I^k(t_i)\right] < \delta$.

Step 2 There exists a game $\widehat{G} = \left\langle \left(\widehat{A_i}\right)_{i=1,2}, (\widehat{g}_i)_{i=1,2} \right\rangle$ such that if an action $\widehat{a}_i \in \widehat{A_i}$ is $0-$rationalizable for some t with $\pi^k(t) \in W$, then $\widehat{a}_i$ is not $0-$rationalizable for any t' with $\pi^k(t') \in Z$.

Step 3 There exists $\gamma > 0$ such that in $\widehat{G}$, if an action $\widehat{a}_i \in \widehat{A_i}$ is $0-$rationalizable for some t with $\pi^k(t) \in W$, then $\widehat{a}_i$ is not $0-$rationalizable for any t' with $\pi^k(t') \in Z$.

Given Step 1 and 2, the proof of Step 3:

The $\widehat{G}$ in Step 2 is a quadratic scoring rule on the first k order beliefs, where k is the one in Step 1. Hence, the ICR actions in this game is determined only by the first k order beliefs, i.e., $R_i\left(s, \widehat{G}, \varepsilon\right) = R_i^k\left(s, \widehat{G}, \varepsilon\right)$ and $h_i^k\left(s|\widehat{a}_i, \widehat{G}\right) = h_i\left(s|\widehat{a}_i, \widehat{G}\right)$ for any $s \subset \mathcal{T}_i$,

$\widehat{a_i} \in \widehat{A_i}$ and $\varepsilon \geq 0$. By Ely-Peski Lemma 16, $h_i^k\left(s|\widehat{a_i},\widehat{G}\right)$ is a continuous function in s. Hence $h_i\left(s|\widehat{a_i},\widehat{G}\right)$ is a continuous function in s too.

Consider $L = \bigcup_{\left\{s\in\mathcal{T}_i:\ \pi^k(s)\in W\right\}} R_i\left(s,\widehat{G},0\right) \subset \widehat{A_i}$, which is finite. For any $\widehat{a_i} \in L$, $h_i\left(s'|\widehat{a_i},\widehat{G}\right) > 0$ for any s' with $\pi^k(s') \in Z$ by Step 2. Since Z is closed (hence compact) and $h_i\left(s|\widehat{a_i},\widehat{G}\right)$ is continuous, there exists $\gamma_{\widehat{a_i}} > 0$ such that $h_i\left(s'|\widehat{a_i},\widehat{G}\right) > \gamma_{\widehat{a_i}}$ for any s' with $\pi^k(s') \in Z$.

Let $\gamma = \min\{\gamma_{\widehat{a_i}} : \widehat{a_i} \in L\}$. Since L is finite and $\gamma_{\widehat{a_i}} > 0$ for any $\widehat{a_i} \in L$ and L is finite, we have $\gamma > 0$. Further $h_i\left(s'|\widehat{a_i},\widehat{G}\right) > \gamma$ for any s' with $\pi^k(s') \in Z$ and any $\widehat{a_i} \in L$. Therefore, Step 3 holds.$\blacksquare$

Given Step 1, the proof of Step 2:

Since W and Z are closed, there is $3\delta > 0$ such that $\rho^k\left[(t)^k,(t')^k\right] > 4\delta$ for any two types t, t' with $\pi^k(t) \in W$ and $\pi^k(t') \in Z$.

We can choose the scoring rule with the grid $A_i = B^1\times...\times B^{k-1}$ such that $\rho^k\left[(s)^k, I^k(s)\right] < \delta$ for any $s \in \mathcal{T}$. Hence, any two types t, t' with $\pi^k(t) \in W$ and $\pi^k(t') \in Z$,

$$\begin{aligned}
\rho^k\left[I^k(t),I^k(t')\right] &\geq \rho^k\left[(t)^k,I^k(t')\right] - \rho^k\left[(t)^k,I^k(t)\right] \\
&\geq \rho^k\left[(t)^k,(t')^k\right] - \rho^k\left[I^k(t'),(t')^k\right] - \rho^k\left[(t'_i)^k,I^k(t'_i)\right] \\
&> 3\delta - \delta - \delta \\
&> \delta.
\end{aligned}$$

Let $B^k \subset \{0, \frac{1}{q}, ..., \frac{q-1}{q}, 1\}^{|B^{k-1}|}$. Choose q such that $\frac{1}{q} < \frac{\delta}{4|B^{k-1}|}$.

Since $\rho^k\left[I^k(t),I^k(t')\right] > \delta$, there is some $E \subset \mathcal{T}^{k-1}$, $I^k(t')[E] > I^k(t)\left[[E]^\delta\right] + \delta \geq I^k(t)[E] + \delta$. Hence,

$$I^k(t')\left[E \cap B^{k-1}\right] > I^k(t')\left[E \cap B^{k-1}\right] + \delta.$$

Then, there is some $b \in \left[E \cap B^{k-1}\right]$ such that $I^k(t')[b] > I^k(t')[b] + \frac{\delta}{\left|E\cap B^{k-1}\right|} \geq I^k(t')[b] + \frac{\delta}{|B^{k-1}|}$. Since $\frac{1}{q} < \frac{\delta}{4|B^{k-1}|}$, t and t' must report different grids. Therefore, any 0-rationalizable grid for t can not be 0-rationalizable for t'.$\blacksquare$

The proof of Step 1:

We prove this by induction.

First, consider $k = 1$. Let $B^1 \subset \{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\} \times \{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\}$. Choose n such that $\frac{1}{n} < \delta$. Hence, for any two adjacent grids $b, b' \in B^1$, $\rho^1(b, b') < \delta$.

Because of the quadratic scoring rule, a type t_i report $b^1(t_i)$ as the most closed grid to $(t_i)^1$. Clearly, $\rho^1((t_i)^1, b^1(t_i)) < \delta$.

We show that $\rho^2[(t_i)^2, I^2(t_i)] < \delta$. For any $E \subset T^{k-1}$,

$$\begin{aligned} I^2(t_i)[E] = I^2(t_i)\left[E \cap B^1\right] &= t_i\left[\{t_{-i} : b^1(t_i) \in E \cap B^1\}\right] \\ &\leq t_i\left[\left\{t_{-i} : t_{-i} \in \left[E \cap B^1\right]^\delta\right\}\right] \\ &\leq t_i\left[\left\{t_{-i} : t_{-i} \in [E]^\delta\right\}\right] \\ &< t_i\left[[E]^\delta\right] + \delta, \end{aligned}$$

where the first inequality follows from $\rho^1((t_{-i})^1, b^1(t_{-i})) < \delta$, hence $\{t_{-i} : b^1(t_i) \in E \cap B^1\} \subset [E \cap B^1]^\delta$. Therefore, $\rho^2[(t_i)^2, I^2(t_i)] < \delta$.

Suppose $\rho^{k-1}\left[(t_i)^{k-1}, b^{k-1}(t_i)\right] < \delta$ for a general k. We show that $\rho^k\left[(t_i)^k, I^k(t_i)\right] < \delta$. For any $E \subset T^{k-1}$,

$$\begin{aligned} I^k(t_i)[E] = I^k(t_i)\left[E \cap B^{k-1}\right] &= t_i\left[\{t_{-i} : b^{k-1}(t_i) \in E \cap B^{k-1}\}\right] \\ &\leq t_i\left[\left\{t_{-i} : t_{-i} \in \left[E \cap B^{k-1}\right]^\delta\right\}\right] \\ &\leq t_i\left[\left\{t_{-i} : t_{-i} \in [E]^\delta\right\}\right] \\ &< t_i\left[[E]^\delta\right] + \delta, \end{aligned}$$

where the first inequality follows from $\rho^{k-1}\left[(t_i)^{k-1}, b^{k-1}(t_i)\right] < \delta$, hence $\{t_{-i} : b^{k-1}(t_i) \in E \cap B^{k-1}\} \subset \left[E \cap B^{k-1}\right]^\delta$. Therefore, $\rho^k\left[(t_i)^k, I^k(t_i)\right] < \delta$.

Suppose $\rho^k\left[(t_i)^k, I^k(t_i)\right] < \delta$ for a general k. We show $\rho^k\left[(t_i)^k, b^k(t_i)\right] < \delta$.

Let $B^k \subset \{0, \frac{1}{h}, ..., \frac{h-1}{h}, 1\}^{|B^{k-1}|}$. Choose h such that $\frac{1}{h} < \frac{\delta}{|B^{k-1}|}$. Hence, for any two adjacent grids $b, b' \in B^k$, $\rho^k(b, b') < \delta$. We show $\rho^k\left[(t_i)^k, b^k(t_i)\right] < \delta$. For any $E \subset T^{k-1}$,

$$\begin{aligned} b^k(t_i)[E] = b^k(t_i)\left[E \cap B^{k-1}\right] &\leq t_i\left[\{t_{-i} : b^{k-1}(t_i) \in E \cap B^{k-1}\}\right] + \frac{\delta}{|B^{k-1}|}\left|E \cap B^{k-1}\right| \\ &\leq t_i\left[\{t_{-i} : t_{-i} \in \left[E \cap B^{k-1}\right]^{\delta}\}\right] + \delta \\ &\leq t_i\left[[E]^{\delta}\right] + \delta, \end{aligned}$$

where the first inequality follows because $b^k(t_i)$ is chosen as the closed grid to $(t_i)^k$. Therefore, $\rho^k\left[(t_i)^k, b^k(t_i)\right] < \delta$.■

References

Dekel, E., D. Fudenberg, and S. Morris (2006): "Topologies on Types," *Theoretical Economics*, 1, 275–309.

Ely, J. C., and M. Peski (2008): "Critical Types," mimeo., Northwestern University.