How common are common-prior types?

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Abstract

We show that in the Merterns-Zamir universal type space the strategic closure of finite common-prior types is nowhere dense under the strategic topology introduced by Dekel, Fudenberg, and Morris (2006). The strategic closure includes all regular types defined in Ely and Peski (2008) and a general notion of common-prior types. Thus, our result is in sharp contrast to Lipman (2003), who shows that under the product topology the set of finite common prior types is dense.

1 Preliminaries

Throughout this paper, for any arbitrary separable metric space Y with metric d_Y, let $\Delta(Y)$ be the space of all probability measures on the Borel σ-algebra of Y endowed with the weak*-topology. It is well known that the weak*-topology is metrizable with the Prohorov distance ρ defined as

$$\rho(\mu, \mu') = \inf\{\gamma > 0 : \mu(E) \leq \mu'(E^\gamma) + \gamma \text{ for every Borel set } E \subseteq Y\}, \forall \mu, \mu' \in \Delta(Y)$$

where $E^\gamma \equiv \{y' : \inf_{y \in E} d_Y(y', y) < \gamma\}$.

For simplicity, assume that there are two players, player 1 and player 2. Given a player $i \in \{1, 2\}$, let $-i$ denote the other player in $\{1, 2\}$. The basic uncertainty is a finite set which is denoted by Θ. By a type space we mean a tuple $(T_i, \pi_i)_{i=1,-2}$ where T_i is an arbitrary set and $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$. We say a type t_i is a finite type if there exists a type space $(T_j, \pi_j)_{j=i,-i}$ such that $t_i \in T_i$ and $T_i \times T_{-i}$ is a finite set.

Let $Y^0 = \Theta$ and $Y^1 = Y^0 \times \Delta(Y^0)$. Then, for $k \geq 2$ define recursively

$$Y^k = \left\{ \left(\theta, \mu^1, ..., \mu^k\right) \in Y^0 \times \Delta\left(Y^0\right) \times \cdots \times \Delta\left(Y^{k-1}\right) : \text{marg}_{Y^{l-2}} \mu^l = \mu^{l-1}, \forall l = 2, ..., k \right\}.$$

Then, the Mertens-Zamir universal type space is defined as

$$T^* = \left\{ \left(\mu^1, \mu^2, ...\right) \in \times_{k=0}^{\infty} \Delta\left(Y^k\right) : \text{marg}_{Y^{l-2}} \mu^l = \mu^{l-1}, \forall l \geq 2 \right\}.$$

For each $k \geq 1$, let $\pi^k : T^* \to \Delta\left(Y^{k-1}\right)$ be the natural projection. For every player i and $k \geq 1$, let T_i^* and Y_i^k denote the copies of T^* and Y^k respectively, write $\pi_i^k : T_i^* \to \Delta\left(Y_{-i}^{k-1}\right)$ for π^k, and define $\mathcal{T}_i^k = \pi_i^k\left(T_i^*\right)$. An element $t_i \in T_i^*$ is a type of player i. For simplicity, we will write t_i^k instead of $\pi_i^k(t_i)$ for the k^{th}-order belief of type t_i.[1] By the result of Mertens and Zamir (1985), T_i^* (endowed with product topology) is homeomorphic to $\Delta\left(\Theta \times T_{-i}^*\right)$. Let π_i^* denote this homeomorphism. In the Mertens-Zamir construction, for any type t_i, the marginal distribution of $\pi_i^*(t_i)$ on Y_{-i}^{k-1} agrees with the distribution t_i^k.

Let ρ^0 be the discrete metric on $Y^0 = \Theta$, i.e., $\rho^0(\theta, \theta') = 1$ if $\theta \neq \theta'$ and $\rho^0(\theta, \theta) = 0$. For $k \geq 1$, let ρ^k be the Prohorov metric on $\Delta\left(Y^{k-1}\right)$ with respect to the metric d^{k-1} on Y^{k-1} defined recursively as

$$d^{k-1}\left[\left(\theta, ..., \mu^{k-1}\right), \left(\theta', ..., \nu^{k-1}\right)\right] = \max\{\rho^0(\theta, \theta'), \rho^{k-1}(\mu^1, \nu^1), ..., \rho^{k-1}(\mu^{k-1}, \nu^{k-1})\}.$$

As defined by Chen, Tillio, Faingold, and Xiong (2009), the uniform-weak topology is generated by the metric

$$d^{uw}(t, s) \equiv \sup_{k \geq 1} \rho^k\left(t^k, s^k\right) \text{ for types } t \text{ and } s \text{ in } T^*.$$

Let $G = (A_i, g_i)_{i=1,2}$ be a finite game where A_i is a finite set of actions for player i and $g_i : A_1 \times A_2 \times \Theta \to [-1, 1]$ is the payoff function. For any $\varepsilon \geq 0$, DFM define the ε-interim-correlated-rationalizable set $R(G, \varepsilon)$ to be the largest (w.r.t. set inclusion)

[1] Note that $\mathcal{T}_i^k = \pi_i^k(\mathcal{T}_i)$ and hence when we write $t_i^k \in \mathcal{T}_i^k$ without specifying the type t_i, t_i^k should be understood as the k^{th}-order belief of some type $t_i \in \mathcal{T}_i$.

set in $\left(\left(2^{A_i}\right)^{T_i^*}\right)_{i=1,2}$ with the best-reply property that for any $i = 1, 2$, $j = 3 - i$, and $a_i \in R_i(t_i, G, \varepsilon)$, there is some $\sigma_{-i} : \Theta \times T_{-i}^* \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \varepsilon) \text{ for all } t_{-i} \in T_{-i}^*;$$

$$\int_{\Theta \times T_{-i}} \left[\sum_{a_{-i} \in A_{-i}} [g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta)] \sigma_{-i}(\theta, t_{-i})[a_{-i}]\right] d\pi_i(t_i) \geq -\varepsilon, \forall a_i' \in A_i.$$

Equivalently, $a_i \in R_i(t_i, G, \varepsilon)$ iff there is some $\nu \in \Delta\left(A_{-i} \times \Theta \times T_{-i}^*\right)$ such that

$$\begin{aligned} \nu[\{(a_{-i}, \theta, t_{-i}) : a_{-i} \in R_{-i}(t_{-i}, G, \varepsilon)\}] &= 1; \\ \text{marg}_{\Theta \times T_{-i}^*} \nu &= \pi_i^*[t_i]; \\ \int_{(a_{-i}, \theta, t_{-i})} [g_i(a_i, a_{-i}, \theta) - g_i(a_i', a_{-i}, \theta)] d\nu &\geq -\varepsilon \text{ for all } a_i' \in A_i. \end{aligned}$$

For each $t_i \in T_i^*$, define $h_i(t_i | a_i, G) \equiv \min\{\varepsilon : a_i \in R_i(t_i, G, \varepsilon)\}$. Let $\beta \in (0, 1)$ and $\mathcal{G}^m$ be the collection of all games in which every player has m actions. For t_i and $t_i' \in T_i^*$, define the strategic distance between t_i and s_i as

$$d^S(t_i, s_i) \equiv \sum_{m=1}^{\infty} \beta^m \sup_{a_i \in A_i^m, G \in \mathcal{G}^m} |h_i(t_i | a_i, G) - h_i(s_i | a_i, G)|;$$

The strategic topology is the topology induced by the metric d^S. It is shown in Chen, Tillio, Faingold, and Xiong (2009) that the uniform-weak topology is stronger than the strategic topology, and they are equivalent around any finite types.

2 Finite CP types are nowhere dense

The following definition of a finite common prior type is standard.

Definition 1 *A type t_i is called a finite Common-prior (CP) type if it is contained in a finite type space (T_i, π_i) and there is a prior μ on $\Theta \times T_i \times T_{-i}$ such that $\mu[\{t_j\}] > 0$ and $\mu(\cdot | t_j) = \pi_j^*(t_j)$ for every $t_j \in T_j$ and every $j \in \{i, -i\}$.*

Observe that every type contained in a finite type space in Definition 1 is also a finite CP type.

The following result have been proven in Dekel, Fudenberg, and Morris (2006) and is reproduced here with an alternative proof. The alternative proof will later be used to show the strategic closure of finite CP types is nowhere dense in the universal type space.

Proposition 1 *Finite CP types are not dense in T_i^* under the strategic topology.*

Proof. We prove this proposition by constructing a finite type t_i^* which is not the uniform-weak limit of any sequence of finite CP types. Since t_i^* is a finite type, by the result of Chen, Tillio, Faingold, and Xiong (2009), strategic convergence to t_i^* implies uniform-weak convergence to t_i^* . Hence, t_i^* is not the strategic limit of any sequence of finite types.

Consider the type space $(\{t_i^*\}, \pi_i)_{i=1,2}$ such that

$$\pi_i(t_i^*)\left[\theta = 0, t_{-i}^*\right] = \pi_{-i}\left(t_{-i}^*\right)\left[\theta = 1, t_i^*\right] = 1.$$

We will show that $d^{uw}(t_i^*, t_i) > 1/4$ for any finite CP type t_i. Suppose instead that there exists a finite CP type t_i and a type space $(T_i, \pi_i; \mu)_{i=1,2}$ such that $\mu(t_i) = \alpha > 0$ and $\mu(\cdot|t_j) = \pi_j(t_j)$ with $\mu-$probability 1, and $d^{uw}(t_i^*, t_i) \leq 1/4$. Since $d^{uw}(t_i^*, t_i) \leq 1/4$ and $\pi_i(t_i^*)[\theta = 0] = 1$, we have

$$\pi_i(t_i)[\theta = 0] = \mu(\theta = 0|t_i) \geq 3/4. \tag{1}$$

Since $d^{uw}(t_i^*, t_i) \leq 1/4$, we can find a set $E^1 (\subset T_{-i})$ such that

$$\pi_i(t_i)\left[E^1\right] = \mu\left(E^1|t_i\right) \geq 3/4 \tag{2}$$

and $d^{uw}\left(t_{-i}^*, t_{-i}\right) \leq 1/4$ for any $t_{-i} \in E^1$.

We claim that $\mu(E^1) \geq 2\alpha$. By Bayes' rule, we have

$$\mu\left(E^1\right) = \frac{\mu(\{t_i\} \times E^1)}{\mu(t_i|E^1)}.$$

Since $\mu(t_i) = \alpha$, by (2) we have $\mu(\{t_i\} \times E^1) \geq 3\alpha/4$. Hence, it remains to show that $\mu(t_i|E^1) \leq 3/8$.

First, since $\pi_{-i}\left(t^*_{-i}\right)[\theta=1]=1$ and $d^{uw}\left(t^*_{-i},t_{-i}\right)\leq 1/4$ for any $t_{-i}\in E^1$, $\pi_{-i}(t_{-i})[\theta=1]=\mu(\theta=1|t_{-i})\geq 3/4$ for every $t_{-i}\in E^1$. Thus, $\mu(\theta=1|E^1)\geq 3/4$. Second, by Bayes' rule,

$$\begin{aligned}
3/4 &\leq \mu\left(\theta=1|E^1\right)\\
&= \mu\left(t_i|E^1\right)\times\mu\left(\theta=1|\{t_i\}\times E^1\right)+\left[1-\mu\left(t_i|E^1\right)\right]\times\mu\left(\theta=1|\left(\neg\{t_i\}\right)\times E^1\right)\\
&\leq \mu\left(t_i|E^1\right)\times\mu\left(\theta=1|\{t_i\}\times E^1\right)+\left[1-\mu\left(t_i|E^1\right)\right]\\
&= 1+\left[\mu\left(\theta=1|\{t_i\}\times E^1\right)-1\right]\times\mu\left(t_i|E^1\right).
\end{aligned}$$

Hence, to show $\mu(t_i|E^1)\leq 3/8$, it suffices to prove $\mu(\theta=1|\{t_i\}\times E^1)\leq 1/3$. By (1) and Bayes' rule, we have

$$\begin{aligned}
3/4 &\leq \mu(\theta=0|t_i)\\
&= \mu\left(E^1|t_i\right)\times\mu\left(\theta=0|\left(\{t_i\}\right)\times E^1\right)+\left[1-\mu\left(E^1|t_i\right)\right]\times\mu\left(\theta=0|\left(\neg\{t_i\}\right)\times E^1\right)\\
&\leq \mu\left(E^1|t_i\right)\times\mu\left(\theta=0|\left(\{t_i\}\right)\times E^1\right)+\left[1-\mu\left(E^1|t_i\right)\right].
\end{aligned}$$

Thus,

$$\frac{\mu(E^1|t_i)-1/4}{\mu(E^1|t_i)}\leq\mu\left(\theta=0|\{t_i\}\times E^1\right).$$

Since $\mu(E^1|t_i)\geq 3/4$ by (2), $\mu(\theta=0|\{t_i\}\times E^1)\geq 2/3$. Hence, $\mu(\theta=1|(\{t_i\})\times E^1)\leq 1/3$.

Recall that $d^{uw}\left(t^*_{-i},t_{-i}\right)\leq 1/4$ for any $t_{-i}\in E^1$. Hence, for each $t_{-i}\in E^1$, we can find a set $E^2_{t_{-i}}\ (\subset T_i)$ such that $\pi_{-i}(t_{-i})\left[E^2_{t_{-i}}\right]=\mu\left(E^2_{t_{-i}}|t_{-i}\right)\geq 3/4$ and $d^{uw}(t^*_i,t'_i)\leq 1/4$ for any $t'_i\in E^2_{t_{-i}}$. Then define $E^2=\cup_{t_{-i}\in E^1}E^2_{t_{-i}}$. Then, $\mu(E^2|E^1)\geq 3/4$. We can similarly prove $\mu(E^2)\geq 2\mu(E^1)\geq 2^2\alpha$. Continuing in this manner, we get for any $k\geq 2$, $\mu\left(E^k\right)>2^k\alpha$. Since $\mu\left(E^k\right)\leq 1$ and $2^k\alpha\to\infty$ as $k\to\infty$, this is a contradiction.■

Next, we prove that the strategic closure of finite CP types are nowhere dense under the strategic topology.

Theorem 1 *The strategic closure of finite CP types is nowhere dense in the universal type space under the strategic topology.*

Proof. Let T^F be the set of all finite CP types and $\overline{T^F}$ be the strategic closure of T^F. By Proposition 1, $T^*\backslash\overline{T^F}$ is a nonempty strategic open set. Let $\bar{t}_i\in T^F$ and t^*_i be the type we construct in Proposition 1. For every integer $n \geq 1$, let $t_i(n) \equiv \left(1 - \frac{1}{n}\right) \bar{t}_i + \frac{1}{n} t_i^*$. We now show that $t_i(n) \in T^* \backslash \overline{T^F}$ by proving $d^{uw}(t_i(n), t_i) > 1/4n$ for all $t_i \in T^F$ and integer $n \geq 2$. Then, since $t_i(n)$ is a finite type, strategic convergence to $t_i(n)$ implies uniform-weak convergence to $t_i(n)$. Hence, $t_i(n) \in T^* \backslash \overline{T^F}$.

Suppose instead that there exists a finite CP type t_i contained in $(T_j, \pi_j, \mu)_{j=i,-i}$ such that $d^{uw}(t_i(n), t_i) \leq 1/4n$ for some $n \geq 1$. Since $\pi_i^*(t_i(n))\left[t_{-i}^*\right] = 1/n$, we can find a set $E^1 (\subset T_{-i})$ such that

$$\pi_i(t_i)\left[E^1\right] = \mu\left(E^1 | t_i\right) \geq 3/4n$$

and $d^{uw}\left(t_{-i}^*, t_{-i}\right) \leq 1/4n$ for any $t_{-i} \in E^1$. That is, for $n \geq 1$, $d^{uw}\left(t_{-i}^*, t_{-i}\right) \leq 1/4$ for some finite CP type $t_{-i} \in E^1$. However, by the proof of Proposition 1, $d^{uw}\left(t_{-i}^*, t_{-i}\right) > 1/4$ for any finite CP type t_{-i}. This is a contradiction.■

3 Strategic closure of finite CP types

3.1 Regular types

Ely and Peski (2008) defined regular types to be types around which product topology is equivalent to the strategic topology. By a result of Lipman (2003), we know that the set of finite CP types is dense in the universal type space under product topology. Hence, the set of regular types is contained in the strategic closure of finite CP types.

3.2 General CP types

We now propose a definition of general CP types and show that finite CP types are dense in the space of CP types in the strategic topology. For any probability measure μ on $\Theta \times T_i \times T_{-i} \subseteq \Theta \times T_i^* \times T_{-i}^*$, we say that μ has *strategic full support* on $T_i \times T_{-i}$ if for every player i, $\mu(G) > 0$ for every strategic open set $G \subseteq T_i^*$ such that $G \cap T_i \neq \varnothing$.

Definition 2 *A type t_i (in the inversal type space) is said to be a Common Prior (CP)* *type if there is a belief-closed subset $T_i \times T_{-i}$ in $T_i^* \times T_{-i}^*$ containing t_i admits a prior μ on $\Theta \times T_i \times T_{-i}$ such that: i) $\mu(\cdot|t_j) = \pi_j^*(t_j)$ for $\mu-$almost surely $t_j \in T_j$ with $j \in \{i, -i\}$; ii) μ has strategic full support in $T_i \times T_{-i}$.*

In Definition 2, condition i) is a standard requirement for a common prior type. In appendix, we show that condition ii) in Definition 2 is necessary to define a general common-prior type.

We now prove that every CP type can be approximated by a sequence of finite CP types under the strategic topology.

Proposition 2 *The set of finite CP types is dense in the space of CP types.*

The proof of Proposition 2 requires the following lemmas:

Lemma 1 *Suppose that $\nu \in \Delta(T_i)$ for some $T_i \subseteq T_i^*$ and that $R_i(t_i, G, \varepsilon) = \overline{A}_i$ for all $t_i \in T_i$. Let $t_i(\nu) \in T_i^*$ be defined as $t_i(\nu) \equiv \pi_i^{*-1}\left[\int \pi_i^*(t_i)\, d\nu[t_i]\right]$. Then, $R_i(t_i(\nu), G, \varepsilon) \supseteq \overline{A}_i$.*

Proof. Suppose $a_i \in \overline{A}_i$. For all $t_i \in T_i$, since $R_i(t_i, G, \varepsilon) = \overline{A}_i$, $a_i \in R_i(t_i, G, \varepsilon)$. Hence, for each $t_i \in T_i$, there is some $\sigma_{-i}^{t_i} : \Theta \times T_{-i}^* \to \Delta(A_{-i})$ such that

$$\text{supp}\sigma_{-i}^{t_i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \varepsilon) \text{ for all } t_{-i} \in T_{-i}^*; \tag{3}$$

$$\int_{\Theta \times T_{-i}^*} \left[\sum_{a_{-i} \in A_{-i}} \left[g_i(\theta, a_i, a_{-i}) - g_i(\theta, a_i', a_{-i})\right] \sigma_{-i}^{t_i}(\theta, t_{-i})[a_{-i}]\right] d\pi_i^*(t_i) \geq -\varepsilon, \forall a_i' \in A_i. \tag{4}$$

Now define

$$\sigma_{-i}(\theta, t_{-i})[a_{-i}] \equiv \int_{t_i \in T_i} \sigma_{-i}^{t_i}(\theta, t_{-i})[a_{-i}]\, d\nu[t_i].$$

It follows from (3) that $\text{supp}\sigma_{-i}(\theta, t_{-i}) \subseteq R_{-i}(t_{-i}, G, \varepsilon)$ for all $t_{-i} \in T_{-i}$.

$$\begin{aligned}
&\int_{\Theta \times T_{-i}^*} \left[\sum_{a_{-i} \in A_{-i}} \left[g_i(\theta, a_i, a_{-i}) - g_i(\theta, a_i', a_{-i})\right] \sigma_{-i}(\theta, t_{-i})[a_{-i}]\right] d\pi_i^*(t_i(\nu)) \\
\geq\ & \int_{T_i} \left[\int_{\Theta \times T_{-i}^*} \left[\sum_{a_{-i} \in A_{-i}} \left[g_i(\theta, a_i, a_{-i}) - g_i(\theta, a_i', a_{-i})\right] \sigma_{-i}^{t_i}(\theta, t_{-i})[a_{-i}]\right] d\pi_i^*(t_i)\right] d\nu[t_i] \\
\geq\ & -\varepsilon.
\end{aligned}$$

■

Lemma 2 *Fix any finite collection of m action games $\overline{\mathcal{G}}^m$, $\gamma' > 0$, and a CP type $\bar{t}_i$ with type space $(T_i, \mu)_{i=1,2}$. There exists a finite CP type space $\left(\widetilde{T}_i, \widetilde{\mu}\right)$ and functions $(f_i)_{i=1,2}$, each $f_i : T_i \to \widetilde{T}_i$, and some $\gamma' > \gamma > 0$ such that $R_i(t_i, G, \varepsilon) \subseteq R_i(f(t_i), G, \varepsilon)$ for μ-almost surely $t_i \in T_i$, $\varepsilon \in \{0, \gamma, 2\gamma, ...\}$, and $G \in \overline{\mathcal{G}}^m$.*

Proof. Following the idea of Dekel, Fudenberg, and Morris (2006), we first define the type space as follows. For any $\gamma > 0$, write $\langle x \rangle^\gamma$ as the smallest number in the set $\{0, \gamma, 2\gamma, ...\}$ greater than x, and let Γ^m be the set of all maps from $A_i^m \times \overline{\mathcal{G}}^m$ into $\{0, \gamma, 2\gamma, ...\}$. We build the type spaces $\left(\widetilde{T}_i, \widetilde{\mu}\right)$ using the subsets of Γ^m by defining $f_i(t_i) \equiv (\langle h_i(t_i|a_i, G)\rangle^\gamma)_{a_i, G}$. Let $\widetilde{T}_i$ be defined such that $\widetilde{t}_i \in \widetilde{T}_i$ iff $\widetilde{t}_i = f_i(t_i)$ for some $t_i \in T_i$ and $\mu\left[f_i^{-1}\left(\widetilde{t}_i\right)\right] > 0$. Note that $\widetilde{T}_i$ is a finite set.

Since μ has strategic full support and $h_i(t_i|a_i, G)$ is continuous in t_i under the strategic topology, $\mu\left[f_i^{-1}(\{f(t_i)\})\right] > 0$ if $t_i \in T_i$ and for all a_i in G with $G \in \overline{\mathcal{G}}^m$, $h_i(t_i|a_i, G) \neq k\gamma$ for some positive integer k. Then, since $A_i^m \times \overline{\mathcal{G}}^m$ is a finite set, we can choose γ such that $\gamma' > \gamma > 0$, $\mu\left[f_i^{-1}(\{f(t_i)\})\right] > 0$ for μ-almost surely $t_i \in T_i$.

Define $\widetilde{\pi}_i : \widetilde{T}_i \to \Delta\left(\Theta \times \widetilde{T}_{-i}\right)$ as follows. For each $\widetilde{t}_i \in \widetilde{T}_i$,

$$\begin{aligned} &\widetilde{\pi}_i\left(\widetilde{t}_i\right)\left[\left\{\widetilde{\theta}, \widetilde{t}_{-i}\right\}\right] \\ = &\int_{f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)} \pi_i^*(t_i)\left[\left\{(\theta, t_{-i}) : \theta = \widetilde{\theta}, f_{-i}(t_{-i}) = \widetilde{t}_{-i}\right\}\right] d\mu\left[t_i | f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)\right]. \end{aligned}$$

Let $\widetilde{\mu} \equiv \mu \circ (I_\Theta \times f_i \times f_{-i})^{-1}$ where I_Θ is the identity mapping on Θ and hence $I_\Theta \times f_i \times f_{-i}$ is a measurable function from $\Theta \times T_i \times T_{-i}$ onto $\Theta \times \widetilde{T}_i \times \widetilde{T}_{-i}$ with $(I_\Theta \times f_i \times f_{-i})(\theta, t_i, t_{-i}) \equiv (\theta, f_i(t_i), f_{-i}(t_{-i}))$.[2] We now establish our claim in three steps:

Step 1 $\widetilde{\mu}\left(\cdot|\widetilde{t}_i\right) = \widetilde{\pi}_i\left(\widetilde{t}_i\right)$ *for $\widetilde{\mu}-$almost surely t_i for $i = 1, 2$, i.e., $\widetilde{\mu}$ is a common prior on $\widetilde{T}_i \times \widetilde{T}_{-i}$.*

[2] It is measurable because $h_i(t_i|a_i, G)$ is a continuous function on T_i under strategic topology which induces the same measurable structure as that induced by the product topology, by a result in Chen, Tillio, Faingold, and Xiong (2009)).

Let $\widetilde{t}_i \in \widetilde{T}_i$. Since $\mu\left[f_i^{-1}\left(\widetilde{t}_i\right)\right] > 0$,

$$\begin{aligned}
\widetilde{\pi}_i\left(\widetilde{t}_i\right)\left[\left\{\widetilde{\theta},\widetilde{t}_{-i}\right\}\right] &= \frac{\int_{f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)} \pi_i^*\left(t_i\right)\left[\left\{\left(\theta,t_{-i}\right):\theta=\widetilde{\theta}, f_{-i}\left(t_{-i}\right)=\widetilde{t}_{-i}\right\}\right] d\mu\left[t_i\right]}{\mu\left[f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)\right]} \\
&= \frac{\int_{f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)} \mu\left[\left\{\left(\theta,t_{-i}\right):\theta=\widetilde{\theta}, f_{-i}\left(t_{-i}\right)=\widetilde{t}_{-i}\right\}|t_i\right] d\mu\left[t_i\right]}{\mu\left[f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)\right]} \\
&= \frac{\mu\left[\left(\theta,t_i,t_{-i}\right):\theta=\widetilde{\theta}, f_i\left(t_i\right)=\widetilde{t}_i, f_{-i}\left(t_{-i}\right)=\widetilde{t}_{-i}\right]}{\mu\left[f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)\right]} \\
&= \frac{\widetilde{\mu}\left[\left\{\left(\widetilde{\theta},\widetilde{t}_i,\widetilde{t}_{-i}\right)\right\}\right]}{\widetilde{\mu}\left[\left\{\widetilde{t}_i\right\}\right]}
\end{aligned}$$

where the second equality is because $\mu\left(\cdot|t_i\right)=\pi_i^*\left(t_i\right)$ for $\mu-$almost surely t_i. Hence, $\widetilde{\mu}\left(\cdot|\widetilde{t}_i\right) = \widetilde{\pi}_i\left(\widetilde{t}_i\right)$ for all $\widetilde{t}_i$.

Step 2 *$R_i\left(t_i,G,\varepsilon\right) \subseteq R_i\left(f_i\left(t_i\right),G,\varepsilon\right)$ for all $t_i \in T_i$ such that $\mu\left[f_i^{-1}\left(\left\{f\left(t_i\right)\right\}\right)\right] > 0$ and $\varepsilon \in \{0,\gamma,2\gamma,...\}$. In particular, $R_i\left(\bar{t}_i,G,\varepsilon\right) \subseteq R_i\left(f_i\left(\bar{t}_i\right),G,\varepsilon\right)$*

For any $\widetilde{t}_i \in \widetilde{T}_i$, we define

$$\eta_i\left(\widetilde{t}_i\right) \equiv \pi_i^{*-1}\left[\int_{f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)} \pi_i^*\left(t_i\right) d\mu\left[t_i|f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)\right]\right].$$

For each $\widetilde{t}_i \in \widetilde{T}_i$, fix any $t_i \in T_i^*$ such that $f_i\left(t_i\right) = \widetilde{t}_i$. Label this type $\zeta_i\left(\widetilde{t}_i\right)$. Since $\varepsilon \in \{0,\gamma,2\gamma,...\}$, $R_i\left(t_i,G,\varepsilon\right) = R_i\left(\zeta_i\left(\widetilde{t}_i\right),G,\varepsilon\right)$ for any $t_i \in T_i$ such that $f_i\left(t_i\right) = \widetilde{t}_i$. By Lemma 1, $R_i\left(\eta_i\left(\widetilde{t}_i\right),G,\varepsilon\right) \supseteq R_i\left(t_i,G,\varepsilon\right)$ for any $t_i \in T_i$ such that $f_i\left(t_i\right) = \widetilde{t}_i$ and in particular $R_i\left(\eta_i\left(\widetilde{t}_i\right),G,\varepsilon\right) \supseteq R_i\left(\zeta_i\left(\widetilde{t}_i\right),G,\varepsilon\right)$.

Define

$$S_i\left(\widetilde{t}_i\right) \equiv R_i\left(\eta_i\left(\widetilde{t}_i\right),G,\varepsilon\right).$$

We then claim that S is an $\varepsilon-$best response set. Suppose $a_i \in S_i\left(\widetilde{t}_i\right)$. Then, there is some $\nu \in \Delta\left(\Theta \times T_{-i} \times A_{-i}\right)$ such that

$$\nu\left[\left(\theta,t_{-i},a_{-i}\right): a_{-i} \in R_{-i}\left(t_{-i},G,\varepsilon\right)\right] = 1; \tag{5}$$

$$\text{marg}_{\Theta\times T_{-i}}\nu = \pi_i^*\left(\eta_i\left(\widetilde{t}_i\right)\right); \tag{6}$$

$$\int\left[g_i\left(\theta,a_i,a_{-i}\right) - g_i\left(\theta,a_i',a_{-i}\right)\right] d\nu \geq -\varepsilon, \forall a_i' \in A_i. \tag{7}$$

We now define $\widetilde{\nu} \in \Delta\left(\Theta \times \widetilde{T}_{-i} \times A_{-i}\right)$ as

$$\widetilde{\nu}\left[\left(\theta, \widetilde{t}_{-i}, a_{-i}\right)\right] \equiv \nu\left[\left(\theta, t_{-i}, a_{-i}\right) : f_{-i}\left(t_{-i}\right) = \widetilde{t}_{-i}\right], \forall\left(\theta, \widetilde{t}_{-i}, a_{-i}\right) \in \Theta \times \widetilde{T}_{-i} \times A_{-i}.$$

First, observe that for almost surely $t_{-i} \in T_{-i}$, $f_{-i}\left(t_{-i}\right) = \widetilde{t}_{-i}$ for some $\widetilde{t}_{-i} \in \widetilde{T}_{-i}$ by the definition of $\widetilde{T}_{-i}$. Moreover, since $R_{-i}\left(\eta_{-i}\left(\widetilde{t}_{-i}\right), G, \varepsilon\right) \supseteq R_{-i}\left(\zeta_{-i}\left(\widetilde{t}_{-i}\right), G, \varepsilon\right) = R_{-i}\left(t_{-i}, G, \varepsilon\right)$ for any $t_{-i} \in T_{-i}$ such that $f_{-i}\left(t_{-i}\right) = \widetilde{t}_{-i}$ and since $S_{-i}\left(\widetilde{t}_{-i}\right)$ is either $R_{-i}\left(\eta_{-i}\left(\widetilde{t}_{-i}\right), G, \varepsilon\right)$ or $R_{-i}\left(\zeta_{-i}\left(\widetilde{t}_{-i}\right), G, \varepsilon\right)$, $a_{-i} \in R_{-i}\left(t_{-i}, G, \varepsilon\right)$ implies $a_{-i} \in S_{-i}\left(f_{-i}\left(t_{-i}\right)\right)$. Hence, (5) implies

$$\widetilde{\nu}\left[\left(\theta, \widetilde{t}_{-i}, a_{-i}\right) : a_{-i} \in S_{-i}\left(\widetilde{t}_{-i}\right)\right] = 1. \tag{8}$$

Second, since $\mu\left[t_i : f_i\left(t_i\right) = \widetilde{t}_i\right] > 0$,

$$\widetilde{\pi}_i\left(\eta_i\left(\widetilde{t}_i\right)\right) = \int_{f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)} \pi_i^*\left(t_i\right) d\mu\left[t_i | f_i^{-1}\left(\left\{\widetilde{t}_i\right\}\right)\right]. \tag{9}$$

Hence, (6) implies

$$\text{marg}_{\Theta \times T_{-i}} \widetilde{\nu} = \widetilde{\pi}_i\left(\widetilde{t}_i\right).$$

Finally, by our construction, $\text{marg}_{\Theta \times A_{-i}} \nu = \text{marg}_{\Theta \times A_{-i}} \widetilde{\nu}$ and hence (7) implies

$$\int \left[g_i\left(\theta, a_i, a_{-i}\right) - g_i\left(\theta, a_i', a_{-i}\right)\right] d\widetilde{\nu} \geq -\varepsilon, \forall a_i' \in A_i. \tag{10}$$

Thus, (8)–(10) altogether imply $a_i \in BR_i\left(S\right)\left(\widetilde{t}_i\right)$.■

Proof of Proposition 2. Lemma 2 is a counterpart of Lemma 12 in DFM. Hence, it follows from the proofs of Lemma 13 and Lemma 8 in DFM that for almost-surely $t_i \in T_i$ and $\varepsilon > 0$, there is some finite CP type $\widetilde{t}_i$ such that $d^S\left(t_i, \widetilde{t}_i\right) < \varepsilon$. Since μ has full strategic support, it follows that for any $t_i \in T_i$ and $\varepsilon > 0$, there is some type t_i' such that $d^S\left(t_i, t_i'\right) < \varepsilon/2$ and $d^S\left(t_i', \widetilde{t}_i\right) < \varepsilon/2$ for some finite CP type $\widetilde{t}_i$. By the triangle inequality for d^S (Dekel, Fudenberg, and Morris, 2006, Lemma 2), $d^S\left(t_i, \widetilde{t}_i\right) < \varepsilon$. This completes the proof.■

A Appendix

Definition 2 *A type t_i (in the inversal type space) is said to be a Common Prior (CP) type if there is a belief-closed subset $T_i \times T_{-i}$ in $T_i^* \times T_{-i}^*$ containing t_i admits a prior μ on*

$\Theta \times T_i \times T_{-i}$ such that: i) $\mu(\cdot|t_j) = \pi_j^*(t_j)$ for $\mu-$almost surely $t_j \in T_j$ with $j \in \{i, -i\}$; ii) μ has strategic full support in $T_i \times T_{-i}$.

We show by examples why condition ii) in Definition 2 is needed to define a general common-prior type. In Appendix A.1, we show why we need a full-support condition; in Appendix A.2, we show why we need the strategic (rather than the product) full-support condition.

A.1 Why a full-support condition is needed?

The following example shows why a full-support condition is needed to define a common-prior type.

Consider a type space $T_i = \{t_i^1, t_i^2\}$ such that $\pi_i^*(t_i^1)\left[t_{-i}^1, \theta = 0\right] = 1$ for $i = 1, 2$; $\pi_2^*(t_2^2)[t_1^2, \theta = 1] = 1$; and

$$\pi_1^*\left(t_1^2\right)\left[t_2^h, \theta = 0\right] = \begin{cases} \alpha, & \text{if } h = 1; \\ 1 - \alpha, & \text{if } h = 2, \end{cases}$$

with α being positive but sufficiently close to 0.

That is, t_i^1 and t_{-i}^1 are types with common knowledge of $[\theta = 0]$, but t_i^2 and t_{-i}^2 are standard non-common-prior types when $\alpha = 0$. When α is positive but sufficiently small, t_1^2 is very close to a non-common-prior type according to our usual sense. However, without the full-support condition, t_1^2 will be viewed as a common-prior type, because it can be generated by the prior which has dirac measure on $[t_1^1, t_2^1, \theta = 0]$. The problem is that, being outside support, t_1^2 can form arbitrary conditional belief. Therefore, we need a full support condition to define a general common-prior type.

A.2 Why the strategic (rather than the product) full support?

The notion of strategic full support ties to game and hence we may wonder whether in defining CP types we can replace strategic full support with product full support, i.e., to require $\mu(G) > 0$ only for every *product* open set $G \subseteq T_i^*$ such that $G \cap T_i \neq \varnothing$. We call these types *product CP types*. However, in the following example, we demonstrate that this modified definition does not capture the intuitive meaning of a common prior, and moreover, Proposition 2 fails to hold.

Example. Let T_i^F be the space of finite CP types and write

$$T_i^F = \bigcup_{n=1}^{\infty} \{t_i \in T_i : (T_i, \mu) \text{ is a finite CP type space and } |T_i \times T_{-i}| = n\}.$$

Clearly, for each finite CP type space $T_i \times T_{-i}$ with cardinality n, $\Delta(\Theta \times T_i \times T_{-i})$ has a countable dense subset under the Euclidean topology on $\Delta(\Theta \times T_i \times T_{-i})$. Hence, there is a countable set M of priors such that for any $t_i \in T^F$ with prior μ, μ is a (Euclidean) limit point of M and each μ in M is associated with a finite CP type space. Let T_i^M be the space of all hierarchies generated by M, i.e., $t_i \in T_i^M$ iff $\pi_i^*(t_i) = \mu(\cdot|t_i)$ for some $\mu \in M$. Using these properties, we can verify that T_i^M is a countable dense subset in T^F in the product topology. Hence, we can write $T_i^M = \{t_i^n\}_{n=1}^{\infty}$ and let μ^n be the prior which generates t_i^n and denote its type space by (T_i^n, μ^n). By the result of Lipman (2003), the product closure of T_i^F is T_i^*. Hence, the product closure of T_i^M is also T_i^*.

For any $k \geq 1$, consider the type space $(T_i, \pi_i)_{i=1,2}$ defined as follows.

$$\begin{aligned} T_1 &= \{t_1(k), t_1^*\} \cup \bigcup_{n=1}^{\infty} T_1^n; \\ T_2 &= \{t_2^*\} \cup \bigcup_{n=1}^{\infty} T_1^n. \end{aligned}$$

Define the prior $\mu^{**} = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu^n$. Define $\pi_i(t_i) = \mu^{**}(\cdot|t_i) = \mu^n(\cdot|t_i)$ for each type $t_i \in T_i^n$ for every n, $\pi_1(t_1^*)[\theta = 0, t_2^*] = \pi_1(t_2^*)[\theta = 1, t_1^*] = 1$, and

$$\pi_1^*(t_1(k)) = \left(1 - \frac{1}{k}\right)\delta_{\{\theta=0,t_2^*\}} + \frac{1}{k}\left(\sum_{n=1}^{\infty}\frac{1}{2^n}\pi_1(t_1^n)\right).$$

We now show that for any k, $t_i(k)$ is a CP type. Observe that every $t_i \in T_i^n$ generates the same hierarchy of beliefs in (T_i^n, μ^n) as in $(T_i, \pi_i)_{i=1,2}$. Hence, we can identify T_i^M in $(T_i, \pi_i)_{i=1,2}$ as in $(T_i^*, \pi_i^*)_{i=1,2}$.

Since product closure of T_i^M is T_i^*, the minimal product closed and belief-closed set in $(T_i^*, \pi_i^*)_{i=1,2}$ containing $t_1(k)$ is the universal type space $T_1^* \times T_2^*$. Since $\mu^{**}[\{t_i^n\}] > 0$ for all n

and the product closure of T_i^M is T_i^*, μ^{**} has product full support. Finally, since $\mu^{**}[\{t_i\}] > 0$ if and only if $t_i \in T_i^n$, $\mu^{**}(\cdot|t_i) = \pi_i^*(t_i)$ for $\mu-$almost surely $t_i \in T_i$ with $i = 1, 2$. Hence, $t_i(k)$ is a CP type. However, we can also verify that $d^{uw}(t_1(k), t_1^*) \leq \frac{1}{k} \to 0$ as $k \to \infty$. Hence, by the result of Chen, Tillio, Faingold, and Xiong (2009), $d^S(t_i(k), t_i^*) \to 0$ as $k \to \infty$. By Proposition 1, t_1^* is not the strategic limit of any sequence of finite CP types. It follows that for some m, $t_1(m)$ is not the strategic limit of any sequence of finite CP types. That is, Proposition 2 fails for some product CP type.■

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