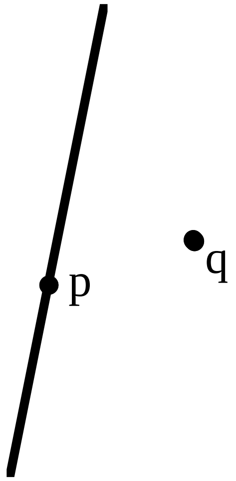
Physical Interpretation of the Interval

- Given two arbitrary events p and q:

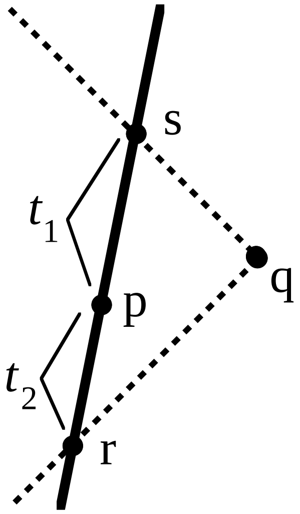
- and a choice of a clock running through p:



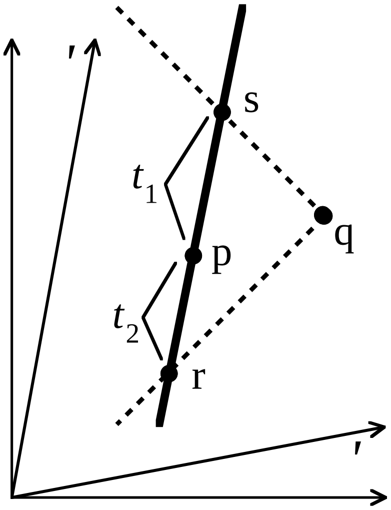
- obtain t_2 and t_1 as follows:

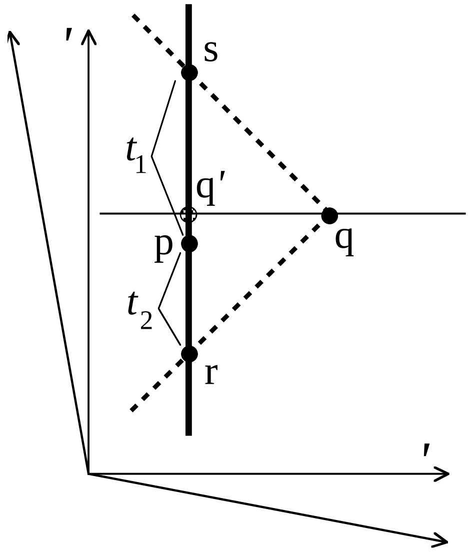
$t_2 = t\ (\mathrm{p}) - t\ (\mathrm{r})$

$t_1 = t\ (\mathrm{s}) - t\ (\mathrm{p})$



- Get a sense of an *apparent* spatial distance Δx between p and q and *apparent* elapsed time Δt, according to *this* clock (i.e. in its rest frame):





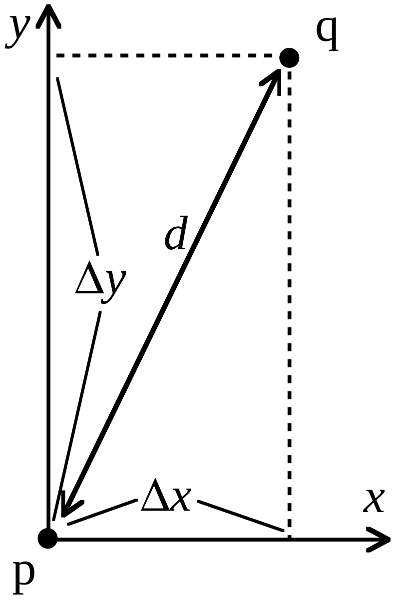
$$\Delta x = 1/2c\,(t_1 + t_2);\ \Delta t = 1/2(t_1 + t_2) - t_2 = 1/2(t_1 - t_2)$$

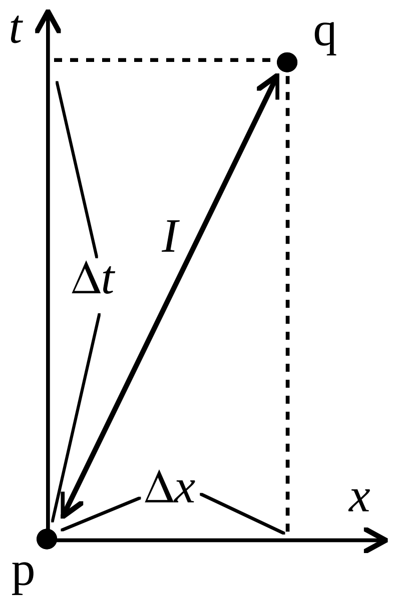
- Conclude that t_1 and t_2 (and hence, Δx and Δt) "encapsulate" features (of the relationship between p and q) intrinsic to ST *and* those associated with a particular choice of clocks.
- Try to "separate out the former" and "clearly post them within ST."
- The Interval $I = t_1 t_2$ depends *only* on the choice of events p and q and *not* on the choice of a clock $\Rightarrow$ intrinsic to ST.
- Different clocks "induce" different perspectives on the relativistic space-time (hence, different Δx's and Δt's), similarly to different Aristotelian observers in the Galilean space-time.

- Expressed in terms of the apparent distance and elapsed time, Δx and Δt:

$$I = t_1 t_2 = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2$$

- Analogy with distance: d^2 vs. I





$$d^2 = (\Delta x)^2 + (\Delta y)^2$$

$$I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2$$

- In a certain (profoundly mathematical) sense, the Interval just IS a (relativistic) distance between events in the relativistic space-time:
- Counterintuitive feature of relativistic “distance”:
    - not Euclidean
    - not positive-definite (may be negative)
    - the distance between two distinct points (i.e., events) may be zero (for lightlike separated events)

Road Atlas Analogy

To specify the intrinsic structure of relativistic space-time, one has to provide:

- A list of all events ("past, present, and future") suitably named or otherwise individuated (without, however, ascribing any coordinatization to them);

- A list of intervals (i.e., relativistic "distances") for all pairs of events.

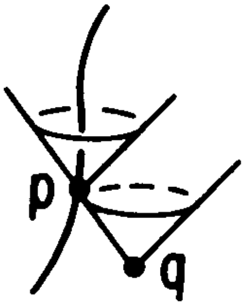
- This information codifies ALL there is to know about the intrinsic structure of ST.

- q and p *timelike* related; q in the future of p, p in the past of q
- $I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 < 0,\ c\Delta t > \Delta x$
- q *inside* the light cone of p

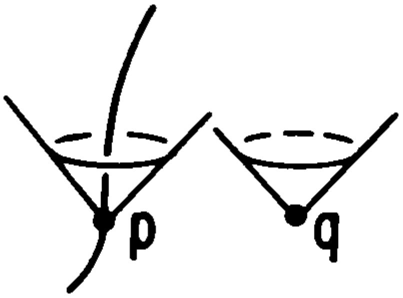
- p and q *timelike* related; p in the future of q, q in the past of p
- $I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 < 0,\ c\Delta t > \Delta x$
- p *inside* the light cone of q

- q and p *lightlike* related; q to the future of p
- $I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 = 0,\ c\Delta t = \Delta x$
- q *on* the light cone of p

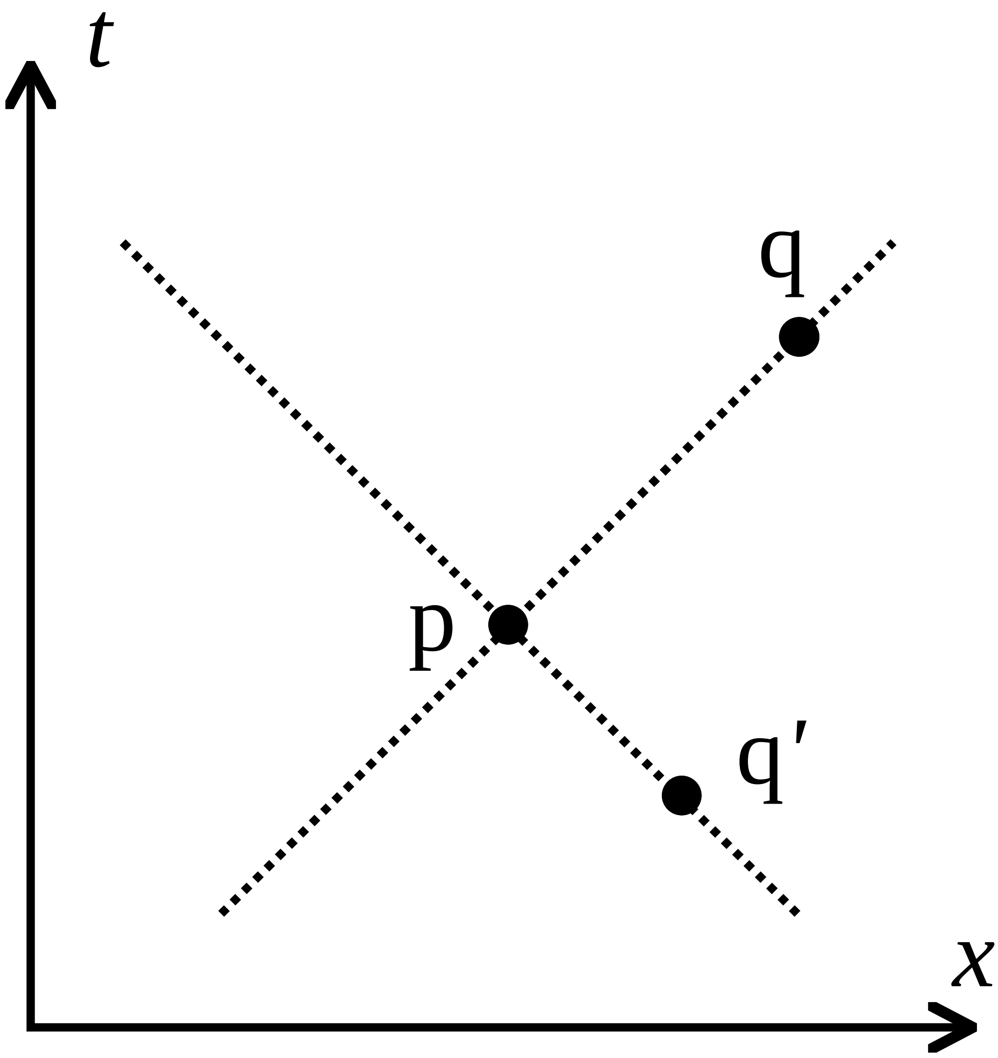
- q and p *lightlike* related; p to the future of q
- $I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 = 0,\ c\Delta t = \Delta x$
- p *on* the light cone of q



- q and p *spacelike* related
- $I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 > 0,\ c\Delta t < \Delta x$
- q and p are *outside* the light cones of each other



Lightlike Relation (Separation)

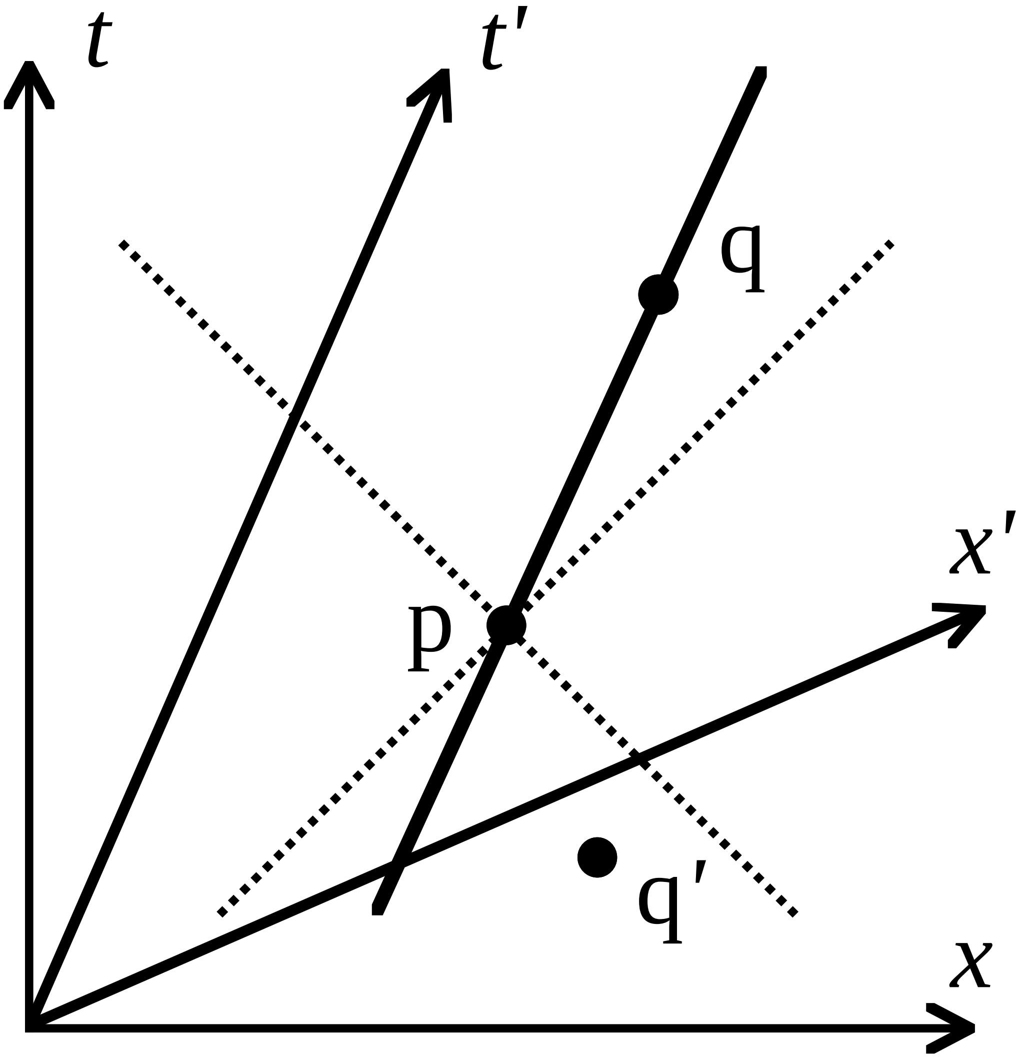


p and q lightlike separated, q to the future of p
p and q′ lightlike separated, q′ to the past of p

$$I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 = 0, \qquad c\Delta t = \Delta x$$

- Light *just makes it* from p to q (and from q′ to p): during the elapsed time between the two events, Δt, light has *just enough* time to travel the distance between the events, Δx: $c\Delta t = \Delta x$

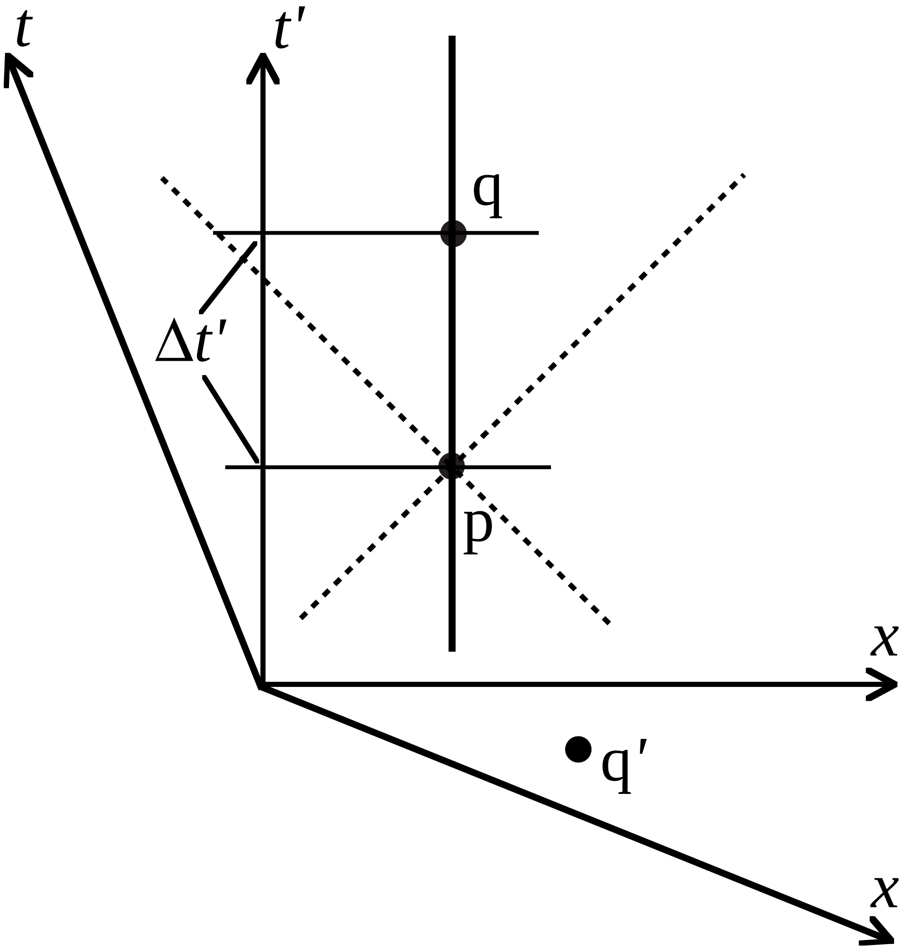
Timelike Relation (Separation)



p and q timelike separated, q to the future of p
p and q′ timelike separated, q′ to the past of p

$$I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 < 0, \qquad c\Delta t > \Delta x$$

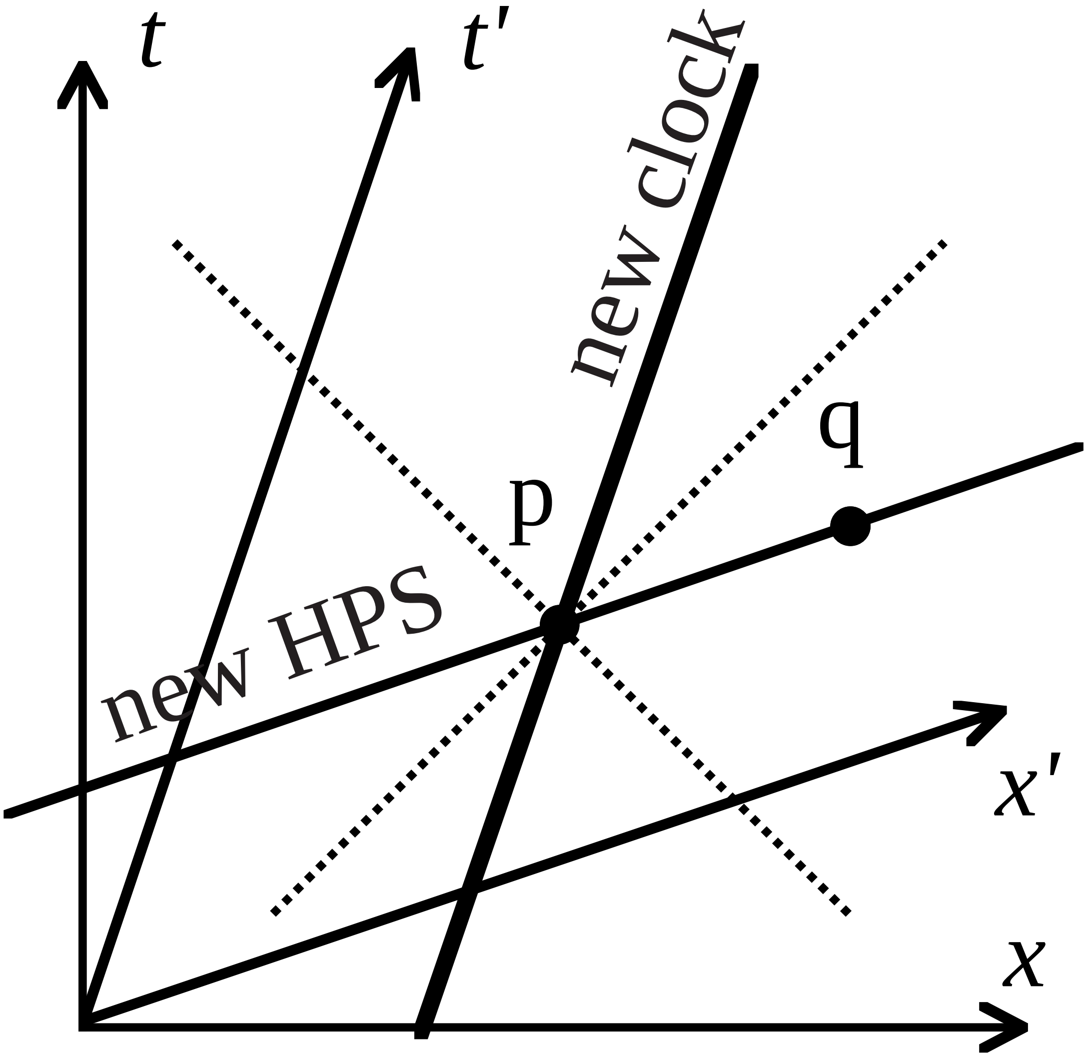
- During the elapsed time between the two events, Δt, light has enough time to travel *farther* than the distance between the events, Δx: $c\Delta t > \Delta x$



- If p and q are timelike related (separated), there is a possible clock running through both p and q.
- According to this clock (i.e., in its rest frame):
    - the spatial distance between p and q is zero: $\Delta x' = 0$;
    - the time elapsed between p and q is: $\Delta t' = \sqrt{-I}$. The negative interval provides a measure of an apparent *time* separation between p and q (in the clock rest frame).

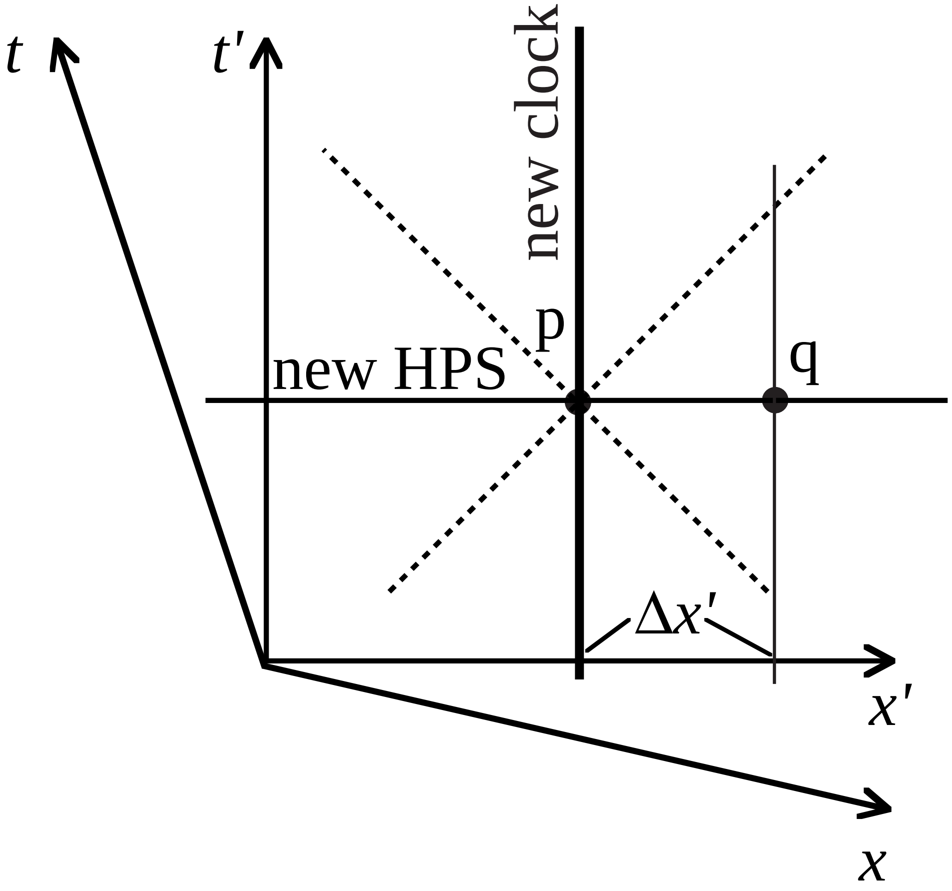
$$I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 = \left(\frac{\Delta x'}{c}\right)^2 - (\Delta t')^2; \qquad \Delta x' = 0$$

Spacelike Relation (Separation)



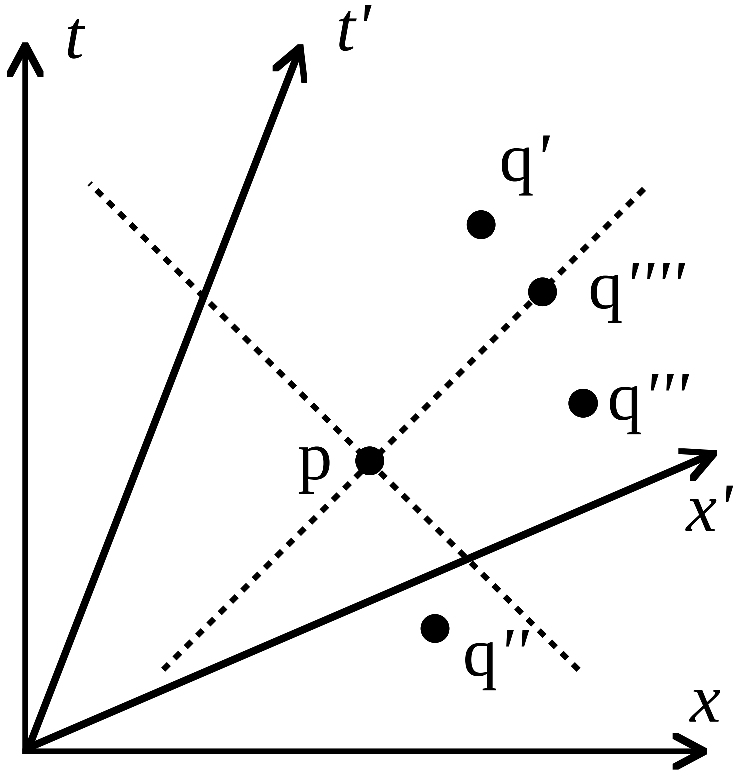
$$I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 > 0, \qquad c\Delta t < \Delta x$$

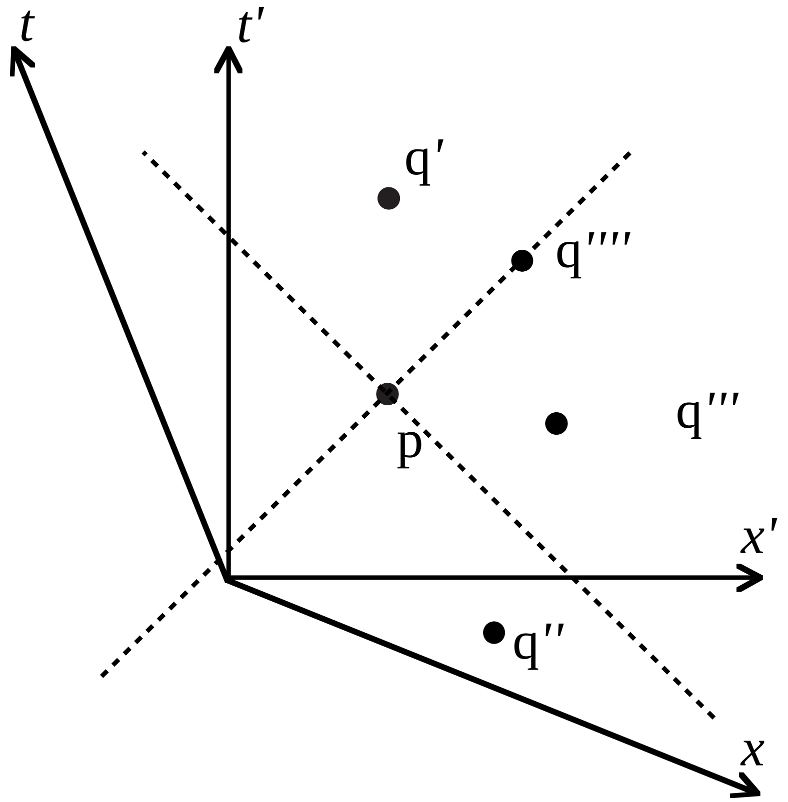
- p and q are too “far apart” in space; light never makes it from p to q; during the elapsed time between the two events, Δt, light does not have enough time to travel the distance between the events, Δx: $c\Delta t < \Delta x$.



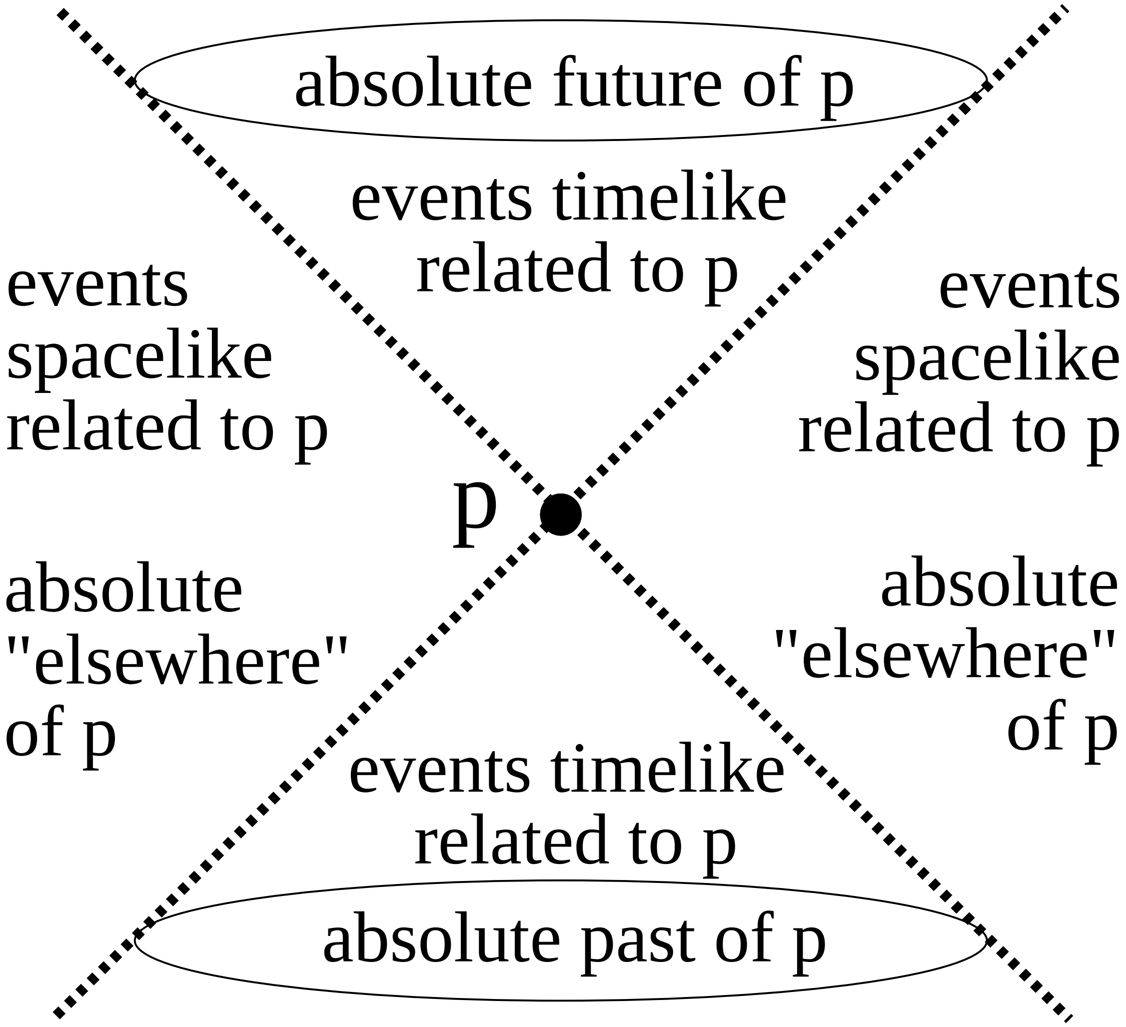
- If p and q are spacelike related, there is *no* clock that could be run through both; but there is a frame in which these events are *simultaneous*.
- According to clock corresponding to that frame:
  - time elapsed between p and q is zero: $\Delta t' = 0$;
  - spatial distance between p and q is: $\Delta x' = c\sqrt{I}$. The positive interval provides a measure of an apparent *spatial* distance between p and q (in the clock rest frame).

$$I = \left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2 = \left(\frac{\Delta x'}{c}\right)^2 - (\Delta t')^2; \qquad \Delta t' = 0$$


t
t′
q′
q′′′′
q′′′
p
x′
q′′
x


t
t′
q′
q′′′′
p
q′′′
x′
q′′
x

Objective Partition of Relativistic Space-Time



absolute future of p

events timelike
related to p

events
spacelike
related to p

events
spacelike
related to p

p

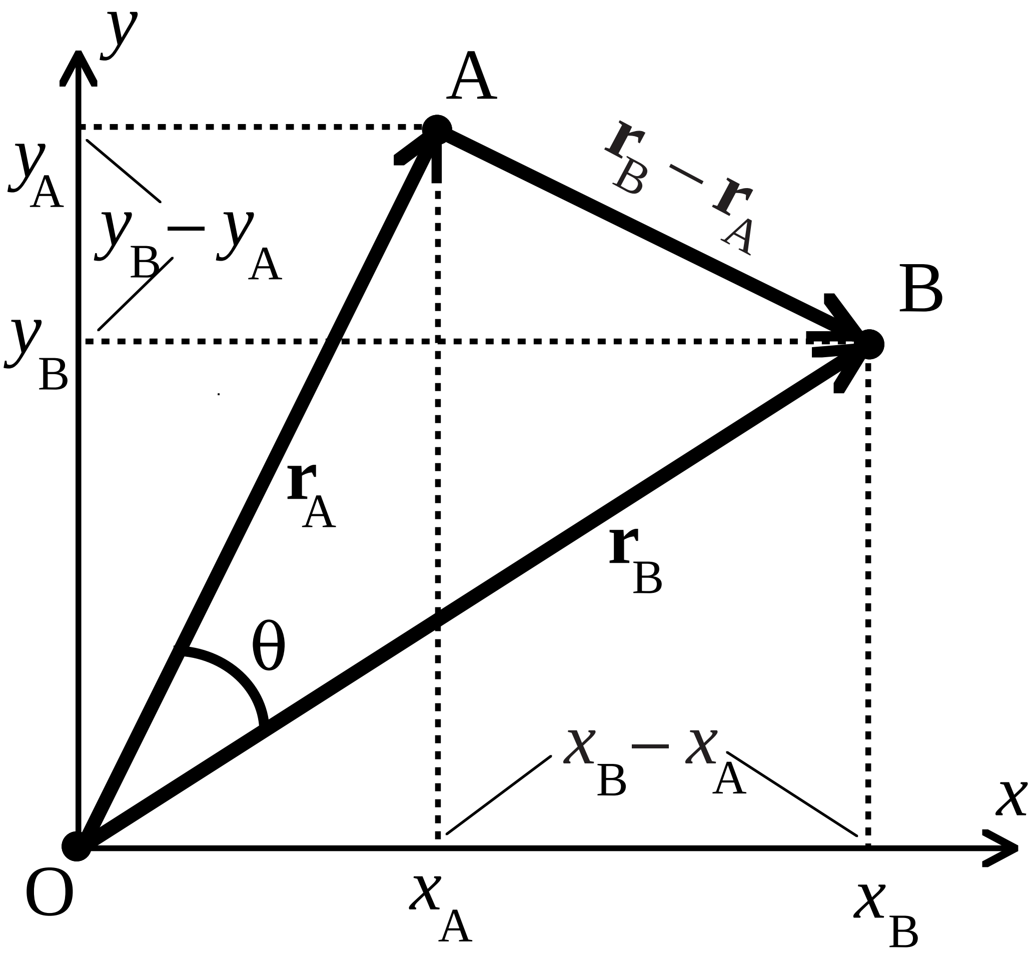
absolute
"elsewhere"
of p

absolute
"elsewhere"
of p

events timelike
related to p

absolute past of p

OPTIONAL: Special Relativistic Space-Time as a *Pseudo-Euclidean Vector Space*.



$\mathbf{r}_A = (x_A, y_A)$

$\mathbf{r}_B = (x_B, y_B)$

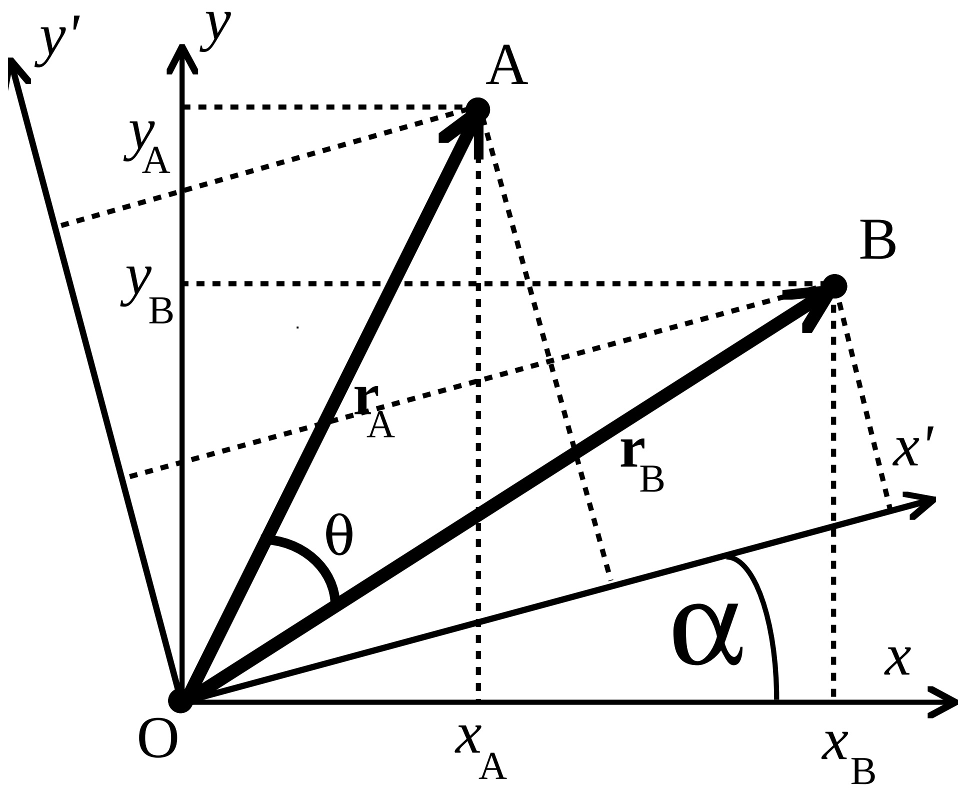
$\mathbf{r}_A + \mathbf{r}_B = (x_A + x_B,\ y_A + y_B)$

$\mathbf{r}_A - \mathbf{r}_B = (x_A - x_B,\ y_A - y_B)$

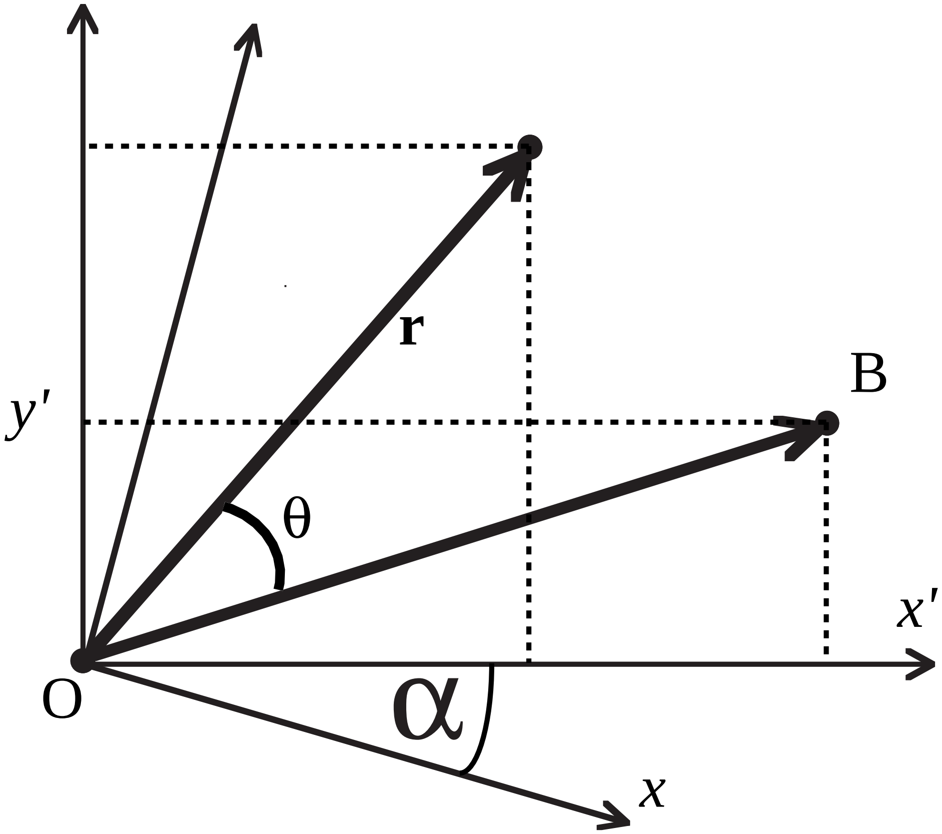
$\mathbf{r}_A\mathbf{r}_B = |\mathbf{r}_A||\mathbf{r}_B| \cos\theta = x_A\, x_B + y_A\, y_B$

$|\mathbf{r}_A| = \sqrt{}(\mathbf{r}_A\mathbf{r}_A) = \sqrt{}(x_A\, x_A + y_A\, y_A) = \sqrt{}(\, x_A{}^2 + y_A{}^2)$

$$\cos\theta = \frac{x_A x_B + y_A y_B}{\sqrt{x_A^2 + y_A^2}\sqrt{x_B^2 + y_B^2}}$$



y'
y
A
y_A
B
y_B
$\mathbf{r}_A$
$\mathbf{r}_B$
x'
θ
α
x
O
x_A
x_B



r
B
y'
θ
x'
O
α
x

$|\mathbf{r}_\mathrm{A}|$, $|\mathbf{r}_\mathrm{B}|$, and θ are preserved under rotation in space.
$|\mathbf{r}_\mathrm{A}| = \sqrt{(x_\mathrm{A}{}^2 + y_\mathrm{A}{}^2)} = \sqrt{(x'_\mathrm{A}{}^2 + y'_\mathrm{A}{}^2)}$

$|\mathbf{r}_\mathrm{A}| = \sqrt{(x_\mathrm{B}{}^2 + y_\mathrm{B}{}^2)} = \sqrt{(x'_\mathrm{B}{}^2 + y'_\mathrm{B}{}^2)}$

Since θ is also preserved, $\mathbf{r}_\mathrm{A}\mathbf{r}_\mathrm{B} = |\mathbf{r}_\mathrm{A}||\mathbf{r}_\mathrm{B}|\cos\theta$ is preserved too.

$\mathbf{r}_\mathrm{A}\mathbf{r}_\mathrm{B} = |\mathbf{r}_\mathrm{A}||\mathbf{r}_\mathrm{B}| \cos\theta = x_\mathrm{A}\, x_\mathrm{B} + y_\mathrm{A}\, y_\mathrm{B} = x'_\mathrm{A}\, x'_\mathrm{B} + y'_\mathrm{A}\, y'_\mathrm{B}$

$(x_\mathrm{A},y_\mathrm{A})$ to $(x'_\mathrm{A},y'_\mathrm{A})$:

$$\begin{aligned} x'_\mathrm{A} &= x_\mathrm{A}\cos\alpha + y_\mathrm{A}\sin\alpha \\ y'_\mathrm{A} &= -x_\mathrm{A}\sin\alpha + y_\mathrm{A}\cos\alpha \end{aligned}$$

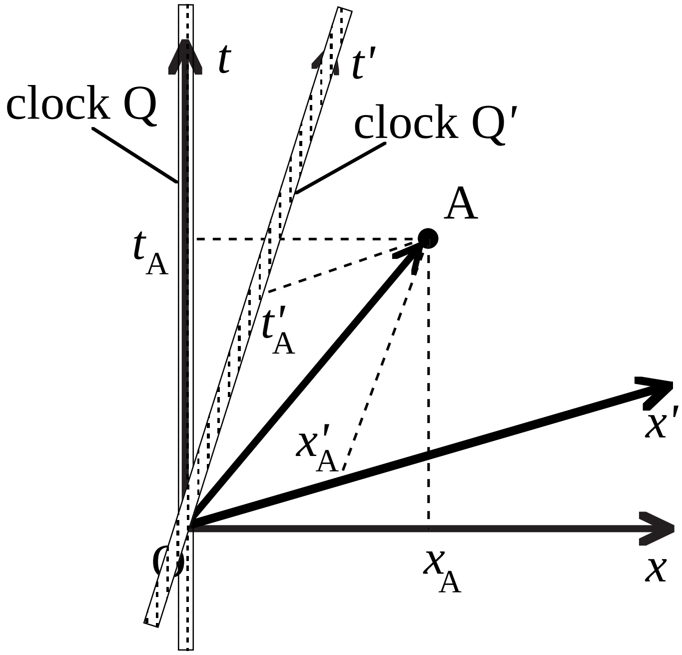
$(x_\mathrm{B},y_\mathrm{B})$ to $(x'_\mathrm{B},y'_\mathrm{B})$:

$$\begin{aligned} x'_\mathrm{B} &= x_\mathrm{B}\cos\alpha + y_\mathrm{B}\sin\alpha \\ y'_\mathrm{B} &= -x_\mathrm{B}\sin\alpha + y_\mathrm{B}\cos\alpha \end{aligned}$$

$\mathbf{r}_\mathrm{A}\mathbf{r}_\mathrm{B} = |\mathbf{r}_\mathrm{A}||\mathbf{r}_\mathrm{B}| \cos\theta = x_\mathrm{A}\, x_\mathrm{B} + y_\mathrm{A}\, y_\mathrm{B}$

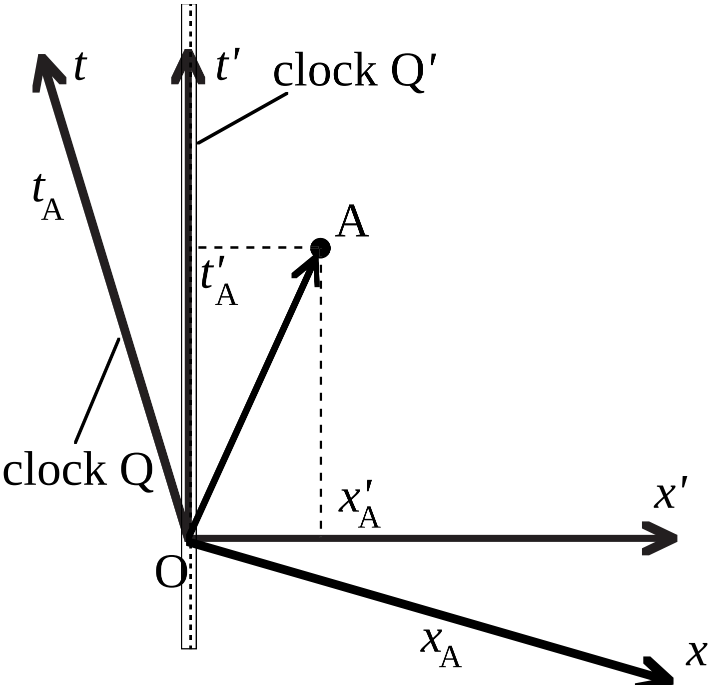
$$= (x_\mathrm{A}, y_\mathrm{A})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_\mathrm{B} \\ y_\mathrm{B} \end{pmatrix}$$

$$\mathbf{r}_\mathrm{A}{}^2 \equiv \mathbf{r}_\mathrm{A}\mathbf{r}_\mathrm{A} = x_\mathrm{A}{}^2 + y_\mathrm{A}{}^2 = (x_\mathrm{A}, y_\mathrm{A})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_\mathrm{A} \\ y_\mathrm{A} \end{pmatrix}$$

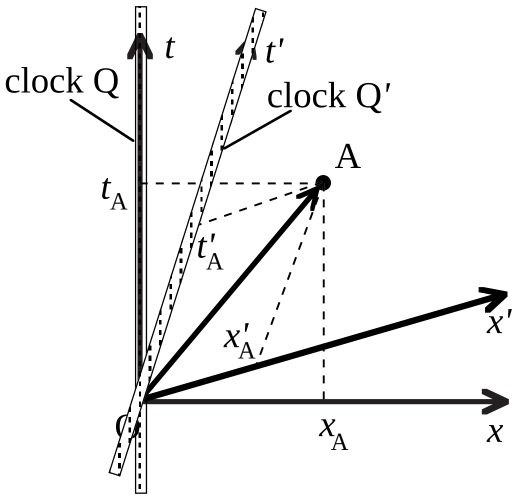


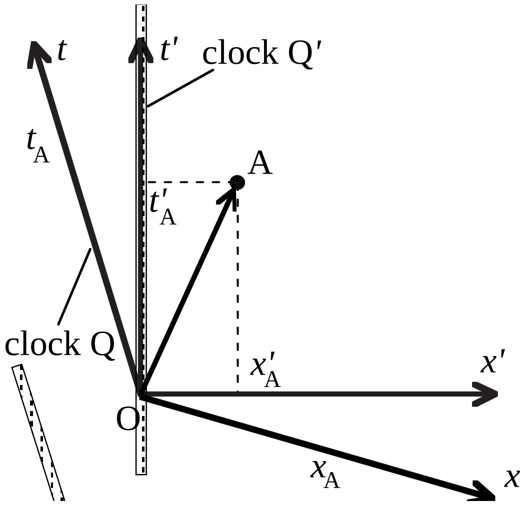
$x_A = \Delta x_A$ — spatial distance between O and A according to clock Q)
$t_A = \Delta t_A$—time elapsed between O and A according to clock Q)

A difference choice of clock (Q′) results in different spatial distance and time elapsed between O and A: $\Delta x'_A$ and $\Delta t'_A$



But the relativistic “rotation” (i.e., transition from Q to Q′ described by Lorentz transfor.) preserves the “length” of vector OA, that is, the Interval between O and A.





$$\boxed{\begin{gathered} I = \left(\frac{\Delta x_A}{c}\right)^2 - (\Delta t_A)^2 = \left(\frac{\Delta x'_A}{c}\right)^2 - (\Delta t'_A)^2 \\ \left(\frac{x_A}{c}\right)^2 - (t_A)^2 = \left(\frac{x'_A}{c}\right)^2 - (t'_A)^2 \end{gathered}}$$

Analogy with scalar product:

$$\mathbf{r}_A{}^2 \equiv \mathbf{r}_A\mathbf{r}_A = x_A{}^2 + y_A{}^2 = (x_A, y_A)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_A \\ y_A \end{pmatrix}$$

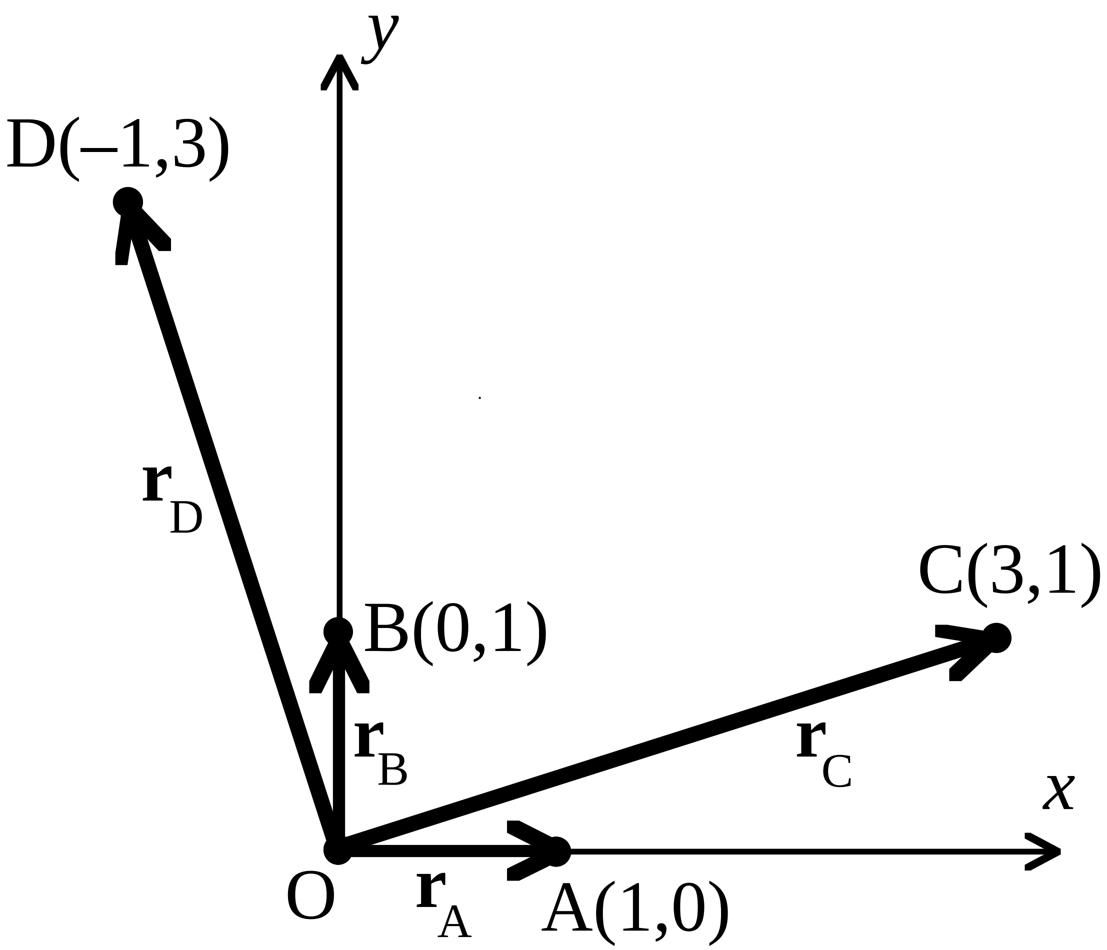
$$I = \left(\frac{x_A}{c}\right)^2 - (t_A)^2 = (x_A, t_A)\begin{pmatrix} 1/c^2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_A \\ t_A \end{pmatrix}$$

$$\mathbf{r}_A\mathbf{r}_B = x_A\, x_B + y_A\, y_B = (x_A, y_A)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_B \\ y_B \end{pmatrix}$$

“Relativistic scalar product” = $\frac{x_A x_B}{c^2} - t_A t_B$

$$= (x_A, t_A)\begin{pmatrix} 1/c^2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_B \\ t_B \end{pmatrix}$$

Orthogonality in Space



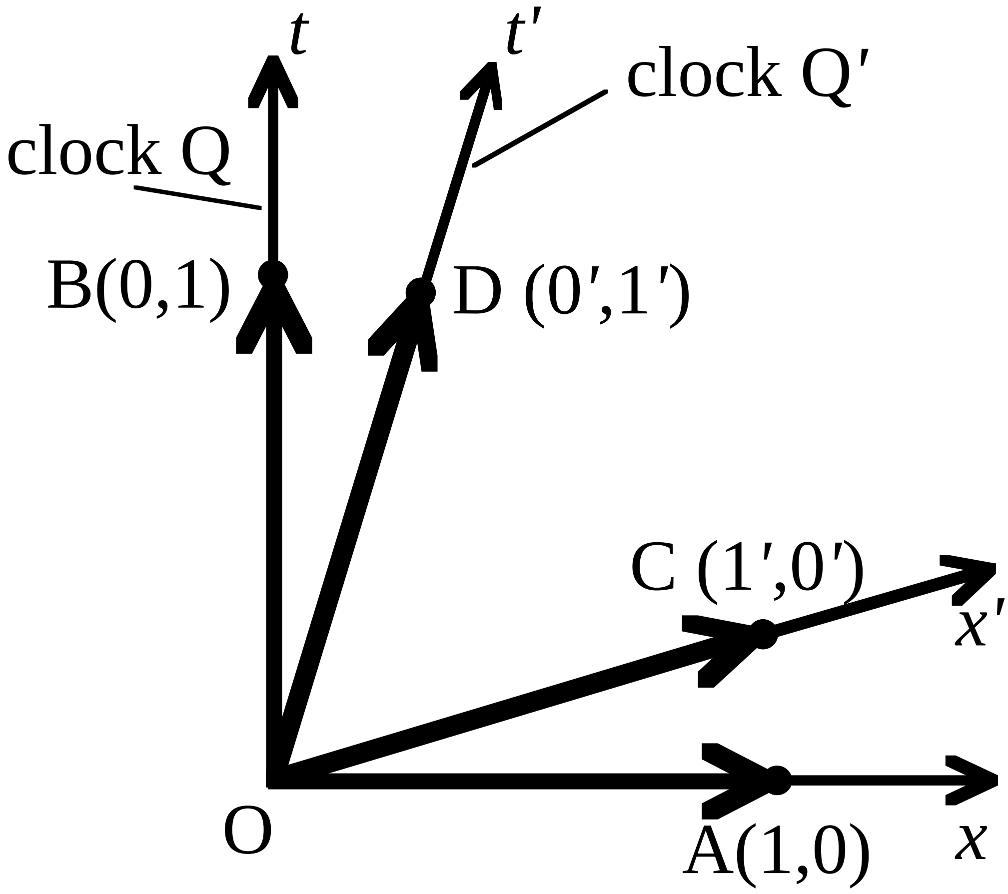
$\mathbf{r}_A \perp \mathbf{r}_B \quad \Leftrightarrow \quad \mathbf{r}_A\mathbf{r}_B = |\mathbf{r}_A||\mathbf{r}_B| \cos\theta = 0 \qquad \Leftrightarrow \quad \cos\theta = 0$
$\quad \Leftrightarrow \quad \theta = 90°$

$\mathbf{r}_A\mathbf{r}_B = x_A\, x_B + y_A\, y_B = 1 \times 0 + 0 \times 1 = 0$

$\mathbf{r}_C \perp \mathbf{r}_D \quad ? \qquad \Leftrightarrow \quad \mathbf{r}_C\, \mathbf{r}_D = 0\, ?$

$\mathbf{r}_C\, \mathbf{r}_D = x_C\, x_D + y_C\, y_D = 3 \times (–1) + 1 \times 3 = –3 + 3 = 0$
$\mathbf{r}_C \perp \mathbf{r}_D$

“Orthogonality” in Relativistic Space-Time

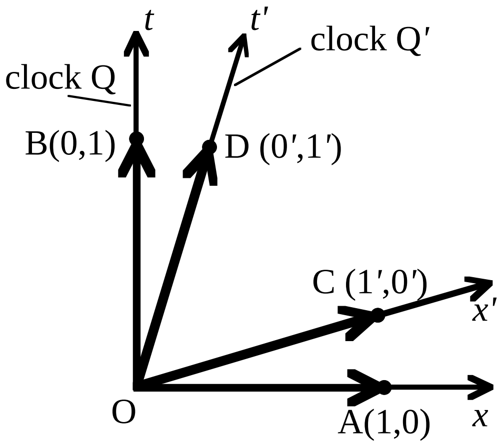


OA ⊥ OB ?

$$\overrightarrow{\mathrm{OA}}\ \ \overrightarrow{\mathrm{OB}} = (x_{\mathrm{A}}, t_{\mathrm{A}})\begin{pmatrix} 1/c^2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{\mathrm{B}} \\ t_{\mathrm{B}} \end{pmatrix} = (1,0)\begin{pmatrix} 1/c^2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1,0)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0 \quad \Rightarrow \quad \mathrm{OA} \perp \mathrm{OB}$$

OC ⊥ OD ?



$x'_{\mathrm{C}} = 1;\ y'_{\mathrm{C}} = 0;\ x'_{\mathrm{D}} = 0;\ y'_{\mathrm{D}} = 1$

From $(x'_{\mathrm{C}}, y'_{\mathrm{C}})$ and $(x'_{\mathrm{D}}, y'_{\mathrm{D}})$ to $(x_{\mathrm{C}}, y_{\mathrm{C}})$ and $(x_{\mathrm{D}}, y_{\mathrm{D}})$:

$$x_{\mathrm{C}} = \gamma(x'_{\mathrm{C}} + ut'_{\mathrm{C}})$$

$$t_{\mathrm{C}} = \gamma(t'_{\mathrm{C}} + \frac{u}{c^2}x'_{\mathrm{C}})$$

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$x_{\mathrm{C}} = \gamma(1 + u \times 0) = \gamma$$

$$t_{\mathrm{C}} = \gamma(0 + \frac{u}{c^2} \times 1) = \frac{\gamma u}{c^2}$$

$$x_{\mathrm{D}} = \gamma(0 + u \times 1) = \gamma u$$

$$t_{\mathrm{D}} = \gamma(1 + \frac{u}{c^2} \times 0) = \gamma$$

$$\overrightarrow{\mathrm{OC}}\ \ \overrightarrow{\mathrm{OD}} = (x_{\mathrm{C}}, t_{\mathrm{C}})\begin{pmatrix} 1/c^2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{\mathrm{D}} \\ t_{\mathrm{D}} \end{pmatrix}$$

$$= (\gamma, \frac{\gamma u}{c^2})\begin{pmatrix} 1/c^2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma u \\ \gamma \end{pmatrix}$$

$$= (\gamma, \frac{\gamma u}{c^2})\begin{pmatrix} \frac{\gamma u}{c^2} \\ -\gamma \end{pmatrix}$$

$$= \gamma \times \frac{\gamma u}{c^2} - \frac{\gamma u}{c^2} \times \gamma = 0$$

OC ⊥ OD! (Yes!)