On Linear Growth Variational Problems Without the Convex Hull Property

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1 Introduction

We are concerned with maximum principles for minimizers of variational integrals

$$I[u] = \int_\Omega f(Du)\,dx, \quad u|\partial\Omega = u_0 \tag{1.1}$$

in a sense which is to made precise below. Here f is a convex function of linear growth and u is a vector valued function of bounded variation. A variational problem is said to have the convex hull property if solutions are contained in closed convex hull of the boundary data. Below we give examples of variational integrals of the type (1.1) for which the associated minimization problem does not have the convex hull property. Nevertheless, we give sufficient conditions on f for minimizers of (1.1) to lie in a closed convex subset of $\mathbf{R}^m$ containing u_0 and conditions for when this set is bounded.

2 Preliminaries and Main Result

A function $u : \mathbf{R}^n \to \mathbf{R}^m$ is said to be of bounded variation, $u \in \mathrm{BV}_{n,m} \equiv \mathrm{BV}(\mathbf{R}^n;\mathbf{R}^m)$, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and if there is a Radon measure $|Du|$ and a $\mathbf{R}^{n\times m}$ valued, $|Du|$-measurable function σ with $|\sigma(x)| = 1$ for $|Du|$-almost every $x \in \mathbf{R}^n$ with the property that for all

$g \in C^\infty_{0,n,m} \equiv C^\infty_0(\mathbf{R}^n;\mathbf{R}^{n\times m})$,

$$\int_{\mathbf{R}^n} \sum_{i=1}^n \sum_{j=1}^m u_i \frac{\partial g_{ij}}{\partial x_j}\, dx = -\int_{\mathbf{R}^n} \sum_{i=1}^n \sum_{j=1}^m g_{ij}\sigma_{ji}\, d|Du|.$$

Sometimes we write $Du/|Du|$ in place of σ. It follows that for $i = 1, \ldots, m$, the coordinate functions u_i lie in $\mathrm{BV}(\mathbf{R}^n)$; there are Radon measures $|Du_i|$ and $\mathbf{R}^n$ valued, $|Du_i|$-measurable functions σ_i with $|\sigma_i(x)| = 1$ for $|Du_i|$-almost every $x \in \mathbb{R}^n$ and $Du_i = \sigma_i \llcorner |Du_i|$. Then for all $g \in C^\infty_{0,n,m}$

$$\int_{\mathbf{R}^n} \sum_{i=1}^n \sum_{j=1}^m g_{ij}\sigma_{ji}\, d|Du| = -\int_{\mathbf{R}^n} \sum_{i=1}^n \sum_{j=1}^m u_i \frac{\partial g_{ij}}{\partial x_j}\, dx = \int_{\mathbf{R}^n} \sum_{i=1}^n \sum_{j=1}^m g_{ij}\sigma_j^{(i)}\, d|Du_j|. \tag{2.1}$$

A real valued function f defined on $\mathbf{R}^{n\times m}$ is of linear growth provided there are positive numbers a, a', b and b' for which

$$a|Q| - b \leq f(Q) \leq a'|Q| + b \quad \forall Q \in \mathbf{R}^{n\times m}.$$

Here $|Q|$ is the Euclidean norm $|Q|^2 = \sum_{i=1}^m \sum_{j=1}^n Q_{ij}^2$. We will occasionally write $f(Q) = f(Q_1, \ldots, Q_m)$ whenever $Q_i \in \mathbf{R}^n$ are the columns of Q of $f(Q) = f(Q_1^T, \ldots, Q_n^T)$ whenever $Q_i \in \mathbf{R}^m$ are the rows of Q. If in addition, f is convex, then there is a homogeneous, convex real valued function f_∞ on $\mathbf{R}^{n\times m}$ defined by the formula

$$f_\infty(Q) = \lim_{t\to\infty} \frac{1}{t} f\left(tQ\right) \quad \forall Q \in \mathbf{R}^{n\times m}.$$

Remark 2.1. In the sequel, f will always denote a real valued convex function of linear growth defined on $\mathbf{R}^{n\times m}$.

Let $u \in \mathrm{BV}_{m,n}$ and u_ϵ be a sequence of smooth functions converging weakly to u in $\mathrm{BV}_{m,n}$. Then

$$\lim_{\epsilon\to 0} \int f(Du^\epsilon)\, dx = \int f(D^a u)\, dx + \int f_\infty(\sigma) d|D^s u|. \tag{2.2}$$

where $D^a u$ is the absolutely continuous with respect to Lebesgue measure part of Du and $D^s u$ is the singular part. The right hand side of (2.2) is the definition of the integral $I[u]$ for a function $u \in \mathrm{BV}_{m,n}$.

If ν and μ are measures with $\nu \ll \mu$, then $D_\mu \nu$ denotes the Radon-Nikodym derivative of ν with respect to ν. The following lemma will be useful shortly.

Lemma 2.2. *Let μ and ν be Radon measures and let ρ and σ be $\mathbf{R}^k$ valued μ-measurable and ν-measurable functions resp. with $|\rho(x)| \le 1$ for μ-almost every $x \in \mathbf{R}^n$ and $|\sigma(x)| = 1$ for ν-almost every $x \in \mathbf{R}^n$. Suppose that $\rho \llcorner \mu$ and $\sigma \llcorner \nu$ are related by the formula*

$$\int_{\mathbf{R}^n} g \cdot \rho \, d\mu = \int_{\mathbf{R}^n} g \cdot \sigma \, d\nu$$

for all $g \in C_0(\mathbf{R}^n; \mathbf{R}^k)$. Then $\nu \ll \mu$ and every ν-measurable set is $\mu \llcorner \{\rho \neq 0\}$ measurable and $\rho = \sigma D_\mu \nu$. If $f \in C(\mathbf{R}^k, \mathbf{R})$, then

$$\int f(\sigma) \, d\nu = \int f(\sigma) D_\mu \nu \, d\mu. \tag{2.3}$$

Proof. The fact that $\nu \ll \mu$ follows easily from the fact that $|\sigma| = 1$ almost everywhere. All μ-measurable sets are then ν-measurable, and a μ-measurable $D_\mu \nu$ exists and is bounded above by 1 for μ-almost every x. By the formula for differentiation of measures, we see that $D_\mu \nu \llcorner \{\rho = 0\} = 0$.

For $r > 0$ define $\lambda(t) = \operatorname{sgn}(t)(r + (|t| - r)_+)$ and let $\rho_r = (\lambda(\rho_1), \ldots, \lambda(\rho_k))$ Then $|\rho_r(x)| \ge r$ for μ-almost every x and $|\rho(x) - \rho_r(x)| \le r$ for every $x \in \{|\sigma| < r\}$ and is zero otherwise.

Without loss of generality, assume that ν and μ are finite. Let $K \subset \{\rho \neq 0\}$ be ν-measurable with $\nu(K) = 0$. Let $B \supset K$ be a Borel set with $\nu(B) = 0$ and $\mu(K) = \mu(B)$. Finally, let C be Borel set with $\mu(C \setminus \{\rho \neq 0\}) = \mu(C \cap \{\rho = 0\}) = 0$ and set $D = B \cap C$.

$$\mu(D \cap \{\rho = 0\}) \le \mu(C \cap \{\rho = 0\}) = 0, \quad \nu(D) \le \nu(B) = 0. \tag{2.4}$$

Let $\{U_i\}_{i=1}^\infty$ a be decreasing sequences of open sets with

$$\nu(D) = \lim_i \nu(U_i), \quad \mu(D) = \lim_i \mu(U_i). \tag{2.5}$$

For a positive integer i and $\epsilon > 0$, there is $g \in C_0(\mathbf{R}^n; \mathbf{R}^k)$ with $\operatorname{spt}(g) \subset W_i$ and $|g| \le 1$ so that

$$\begin{aligned} \mu(U_i) &\le \epsilon + \frac{1}{r} \int_{U_i} g \cdot \rho_r d\mu = \epsilon + \frac{1}{r} \left(\int_{U_i} g \cdot \sigma d\nu + \int_{U_i} g \cdot (\rho_r - \rho) d\mu \right) \\ &\le \epsilon + \frac{1}{r} \nu(U_i) + \mu(U_i \cap \{|\rho| < r\}). \end{aligned}$$

As ϵ was arbitrary, we have $\mu(U_i) \leq r^{-1}\nu(U_i) + \mu(U_i \cap \{|\sigma| < r\})$. Recall D is Borel. Thus $D \cap \{|\rho| < r\}$ and $W_i \cap \{|\rho| < r\}$ are μ-measurable. Thus, from (2.5) and (2.4), letting $i \to \infty$ we find

$$\mu(D) \leq r^{-1}\nu(D) + \mu(D \cap \{|\rho| < r\}) = \mu(D \cap \{|\rho| < r\}).$$

Finally, $D \cap \{|\rho| = 0\}$ and $D \cap \{|\rho| < r\}$ are μ-measurable for each r. Letting $r \to 0$ gives

$$\mu(D) \leq \lim_{r \to 0} \mu(D \cap \{|\rho| < r\}) = \mu(D \cap \{|\rho| = 0\}) = 0.$$

Consequently, D and hence K have μ-measure zero. This implies that $\mu \llcorner \{\rho \neq 0\} \ll \nu \llcorner \{\rho \neq 0\}$. However, $\nu \llcorner \{\rho = 0\} = 0$ and hence $\mu \llcorner \{\rho \neq 0\} \ll \nu$. This proves the first claim.

For $t > 0$, $(f(\sigma)D_\mu\nu)^{-1}([t,\infty))$ is a ν-measurable subset of $\{\rho \neq 0\}$ and is hence μ-measurable. On the other hand, $(f(\sigma)D_\mu\nu)^{-1}(\{0\})$ is the union of a ν-measurable subset of $\{\rho \neq 0\}$ with $\{\rho = 0\}$ and is hence μ-measurable. Thus $f(\sigma)D_\mu\nu$ is μ-measurable and the second claim follows from the Radon-Nikodym theorem. □

Corollary 2.3. *Suppose that* $u, v \in \mathrm{BV}_{n,m}$ *and for all* $g \in C_{0,n,m}$,

$$\int_{\mathbf{R}^n} \sum_{i}^{n} \sum_{j,k=1}^{m} g_{ij} \Lambda_{kj} \left(\frac{Du}{|Du|}\right)_{ik} d|Du| = \int_{\mathbf{R}^n} \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} \left(\frac{Dv}{|Dv|}\right)_{ji} d|Dv|$$

for some $\Lambda \in L^\infty(|Du|; \mathbf{R}^{m \times m})$. *Then*

$$\int f_\infty \left(\frac{Dv}{|Dv|}\right) d|D^s v| = \int f_\infty \left(\frac{Du}{|Du|}\Lambda\right) d|D^s u|. \tag{2.6}$$

Proof. Without loss of generality, $\|\Lambda\|_{L^\infty(|Du|)} \leq 1$. Let

$$\mu = |D^s u|, \quad \nu = |D^s v|, \quad \rho = \frac{Du}{|Du|}\Lambda, \quad \sigma = \frac{Dv}{|Dv|}.$$

Applying lemma 2.2, $\rho = \sigma D_{|D^s u|}|D^s v|$, i.e.

$$\frac{Du}{|Du|}\Lambda = \frac{Dv}{|Dv|} D_{|D^s u|}|D^s v|.$$

Using the change of variables formula (2.3), the homogeneity of f_∞ immediately gives (2.6).

□

A form of (2.6) particularly useful for our purposes goes as follows. Let $u \in \mathrm{BV}_{n,m}$ with $Du = \tau \llcorner |Du|$ and for $i \in \{1, \ldots, m\}$, the coordinate functions u_i satisfy $Du_i = \tau_i \llcorner |Du_i|$. Similarly, suppose that $v \in \mathrm{BV}_{n,m}$ and the coordinate functions of v_i and u_i are related by $Dv_i = \eta_i Du_i$ for some scalar valued functions $\eta_i \in L^\infty(|Du_i|) \subset L^\infty(|Du|)$. Then, if $Dv = \sigma \llcorner |Dv|$ and $Dv_i = \sigma_i \llcorner |Dv_i|$, lemma 2.2 applied to (2.1) implies

$$\eta_i \tau_i^{(j)} = \sigma_i^{(j)} D_{|Du_i|}|Dv_i|, \quad \sigma_{ji} = \sigma_i^{(j)} D_{|Dv|}|Dv_i|, \quad \tau_{ji} = \tau_i^{(j)} D_{|Du|}|Du_i|.$$

Multiplying the last equation by η_i along with the fact that $|Dv_i| \ll |Du_i| \ll |Du|, |Dv_i| \ll |Dv|$ gives

$$\eta_i \tau_{ji} = \eta_i \tau_i^{(j)} D_{|Du|}|Du_i| = \sigma_i^{(j)} D_{|Du_i|}|Dv_i| D_{|Du|}|Du_i| = \sigma_i^{(j)} D_{|Du|}|Dv_i| = \sigma_{ji} D_{|Du|}|Dv|.$$

Again, if $\Lambda\tau = \sigma D_{|Du|}|Dv|$ where Λ is as in lemma (2.3), then the above equations shows that

$$\Lambda_{ij} = \eta_i \delta_{ij}, \quad i, j \in \{1, \ldots, m\}. \tag{2.7}$$

Corollary 2.4. *If $u \in \mathrm{BV}_{n,m}$, $R \in O(m)$ and $v = Ru$, then $Dv = DuR^T$ and $|Dv| = |Du|$.*

Proof. Clearly

$$\int_{\mathbf{R}^n} \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} \left(\frac{Dv}{|Dv|} \right)_{ji} d|Dv| = \int_{\mathbf{R}^n} \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} R_{ik} \left(\frac{Du}{|Du|} \right)_{jk} d|Du|.$$

Set $\mu = |Du|, \nu = |Dv|, \sigma = Du/|Du|R^T$ and $\rho = Dv/|Dv|$. Then $|\sigma(x)| = 1$ for μ-almost every $x \in \mathbf{R}^n$ and $|\rho(x)| = 1$ for ν-almost every $x \in \mathbf{R}^n$. Lemma 2.2 then implies that $|Du| \ll |Dv|$ and $|Dv| \ll |Du|$ and $\rho = \sigma D_\mu \nu$. However, by the definition of $|Dv|$ and $|Du|$ as the variation measure associated with the distributional derivatives Du and Dv resp., $|Du|(U) = |Dv|(U)$ for any open set U and hence $D_\mu \nu = 1$. $\square$

In the sequel, Ω is a bounded Lipschitz domain in $\mathbf{R}^n$, $\tilde{\Omega}$ is an open ball containing the closure of Ω and u_0 is an element of the Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n; \mathbf{R}^m)$. The class of admissible functions is

$$\mathscr{M} \equiv \{v \in \mathrm{BV}_{n,m}; v(x) = u_0(x) \text{ for } \mathscr{L}^n\text{-almost every } x \in \tilde{\Omega} \setminus \Omega\}.$$

For $u \in \mathscr{M}$ and f a convex, linear growth function, the appropriate variational integral of type (1.1) for functions of bounded variation with boundary data u_0 is

$$I[u] \equiv \int_{\tilde{\Omega}} f(Du)$$

as defined in (2.2). $I[\cdot]$ is lower semicontinuous with respect to BV-weak convergence in $\mathscr{M}$ and a solution to

$$u \in \mathscr{M}, \quad I[u] = \min\{I[v] : v \in \mathscr{M}\} \tag{2.8}$$

exists. If $z \in \mathbf{S}^{m-1}$ and $q \in \mathbf{R}^m$, then define $P_z = I - z \otimes z \in \mathbb{R}^{m \times m}$ and

$$K_{q,z} \equiv \{x \in \mathbf{R}^m : (x - q) \cdot z \leq 0\}.$$

Theorem 2.5. *Suppose that* $u_0(\tilde{\Omega}) \subset K_{q,z}$ *and* u *satisfies* (2.8). *If there is* $z \in \mathbf{S}^{m-1}$ *with the property that* $f(QP_z) \leq f(Q)$ *and* $f_\infty(QP_z) < f_\infty(Q)$ *for all* $Q \neq QP_z$. *Then there is* $w \in W^{1,1}(\tilde{\Omega})$ *with*

$$z \cdot u(x) = w(x) \text{ whenever } u(x) \in \mathbf{R}^m \setminus K_{q,z}.$$

If in addition, $f(QP_z) < f(Q)$ *for all* $Q \neq QP_z$, *then* $u(x) \in K_{q,z}$ *for* $\mathscr{L}^n$*-almost every* $x \in \tilde{\Omega}$.

Example 2.6. If f has the form $f(Q) = h(|Q|)$ for some strictly increasing function h, then the first hypothesis of proposition 2.5 is satisfied for all $z \in \mathbf{R}^m$. For such f, problem (2.8) then has the convex hull property.

If f has the form

$$f(Q) = \left(|Q_1|^2 + \cdots + |Q_m|^2 + Q_{11}Q_{12}\right)^{1/2},$$

then the first hypothesis proposition 2.5 is satisfied for all $z = (1, 1, \dots)$ and $z = (-1, 1, \dots)$ but not $z = (1, 0, \dots)$ or $z = (0, 1, \dots)$. For such f, problem (2.8) does not have the convex hull property (see proposition 3.3.)

Proof. We may assume without loss of generality that f is nonnegative, $z = e_m$ and $q = 0$. To see this, suppose u satisfies (2.8) and choose $R \in O(n)$ with $Rz = e_m$ and let $\tilde{u}_0 = R(u_0 - q)$

and let $\tilde{\mathscr{M}}$ be the admissible class with data $\tilde{u}_0$. Then $\tilde{u}_0(\tilde{\Omega}) \subset K_{0,e_m} \equiv \{x \in \mathbf{R}^m : x_m \leq 0\}$. We claim that $\tilde{u}(x) \in K_{0,e_m}$ for $\mathscr{L}^n$-almost every $x \in \tilde{\Omega}$. Let $\tilde{u} = R(u-q)$ so that $\tilde{u}(x) \in \tilde{\mathscr{M}}$. If $\tilde{v} \in \tilde{\mathscr{M}}$, then corollary 2.4 implies $\tilde{I}[\tilde{v}] = I[v] + \mathscr{L}^n(\tilde{\Omega}) \min f$ where $\tilde{I}[\cdot]$ is defined as in (2.2) by the linear growth function $\tilde{f}(Q) \equiv f(QR) + \min f$. Then, by the minimality of u,

$$\tilde{I}[\tilde{u}] = I[u] + \mathscr{L}^n(\tilde{\Omega}) \min f \leq I[v] + \mathscr{L}^n(\tilde{\Omega}) \min f = \tilde{I}[\tilde{v}].$$

Note that $e_m \otimes e_m R = Rz \otimes z$ and thus $P_{e_m} R = RP_z$. Applying this fact to the function $\tilde{f}$, we learn

$$\tilde{f}(QP_{e_m}) = f(QP_{e_m}R) = f(QRP_z) < f(QR) = \tilde{f}(Q)$$

if $QP_{e_m}R = QRP_z \neq QR$, i.e. $QP_{e_m} \neq Q$. Applying the remainder of the proof to $\tilde{u}$, the claim follows. Since $\{\tilde{u} \in K_{0,e_m}\} = \{u \in K_{q,z}\}$, the claim then implies the proposition.

Let $0 < \alpha < 1$. According to [HZ94] lemma 1.2, there is nonnegative, $|Du_m|$-measurable function η with $\eta(x) \leq 1$ for $|Du_m|$-almost every $x \in \mathbf{R}^n$ with the property that $D(u_m)_+ = \eta Du_m$. Define a function $v \in \mathrm{BV}_{n,m}$ by

$$v = (v_1, \ldots, v_m) \equiv (u_1, \ldots, u_{m-1}, \min\{u_m, 0\} + \alpha \max\{u_m, 0\}).$$

Then, if $\beta = 1 - \alpha$,

$$Dv_m = (1 - \beta\eta) Du_m, \quad Dv_i = Du_i, \quad i = 1, \ldots, m-1.$$

Let $Du = \tau \llcorner |Du|, Dv = \sigma \llcorner |Dv|$. From the representation formula (2.7), we have that $\sigma \llcorner |Dv| = \tau\Lambda \llcorner |Du|$ as bounded linear functionals on $C_{0,n,m}$ where $\Lambda \in L^\infty(|Du|; \mathbf{R}^{m\times m})$ is diagonal with all nonzero entries equal to 1, except the last, $\Lambda_{mm} = (1 - \beta\eta)$. Then, applying (2.6) to the second integral below,

$$I[v] = \int f(D^a v)\, dx + \int f_\infty(\sigma) d|D^s v| = \int f(D^a u\Lambda)\, dx + \int f_\infty(\tau\Lambda) d|D^s u|.$$

Note that for $|Du|$-almost every $x \in \mathbf{R}^n$, $0 \leq \beta\eta(x) \leq 1$ that $\Lambda(x) = (1 - \beta\eta(x))I + \beta\eta(x)P_z$. By the convexity of f and f_∞,

$$I[v] \leq \int (1-\beta\eta) f(D^a u) + \beta\eta f(D^a u P_z)\, dx + \int (1-\beta\eta) f_\infty(\tau) + \beta\eta f_\infty(\tau P_z)\, d|D^s u|.$$

As $f(DuP_z) \leq f(Du)$, we see by the minimality of $I[u]$ that $I[v] \leq I[u] \leq I[v]$. It follows that the second inequality must be equality and hence subtracting and dividing by $\beta \neq 0$ we have

$$\int \eta f(D^a u P_z)\,dx + \int \eta f_\infty(\tau P_z)\,d|D^s u| = \int \eta f(D^a u)\,dx + \int \eta f_\infty(\tau)\,d|D^s u|.$$

Since the integrands on the right hand side majorize the integrands on the left hand side respectively, it must be that

$$\int \eta(f(D^a u P_z) - f(D^a u))\,dx = 0, \qquad \int \eta(f_\infty(\tau P_z) - f_\infty(\tau))\,d|D^s u| = 0.$$

Since η is nonnegative $|Du_m|$-almost everywhere, the above two integrands are $|Du_m|$-almost everywhere non-positive, and hence

$$\eta(f(D^a u P_z) - f(D^a u)) = \eta(f_\infty(\tau P_z) - f_\infty(\tau)) = 0 \quad |Du_m|\text{-almost everywhere.} \tag{2.9}$$

Let

$$E = \{\eta\tau_m \neq 0\} \cup \{\eta D^a u_m \neq 0\},$$
$$F = \{\eta(f(\tau P_z) - f(\tau)) < 0\} \cup \{\eta(f(D^a u P_z) - f(D^a u)) < 0\}.$$

E and F are $|Du_m|$-measurable and $F \subset E$. By (2.9), $|Du_m|(F) = 0$. Furthermore, $\{\eta\tau_m \neq 0\} \subset F$ and thus $|Du_m|(\{\eta\tau_m \neq 0\}) \leq |Du_m|(F) = 0$. This implies that

$$D^s(u_m)_+ = \eta\tau_m |D^s u| = 0.$$

Then, $(u_m)_+ \in W^{1,1}(\tilde{\Omega})$.

If $f(QP_z) < f(Q)$ and $f_\infty(QP_z) < f_\infty(Q)$whenever $QP_z \neq Q$, then $E \subset F$. In this case,

$$|Du_m|(E) = |Du_m|(F) = 0.$$

This proves the second claim of the theorem because

$$D(u_m)_+ = \eta D u_m = 0$$

implying that $\mathscr{L}^n(\{x : u_m(x) \geq 0\}) = 0$ (by the Poincaré inequality for example.)

$\square$

Corollary 2.7. *Suppose that* $f : \mathbf{R}^n \to \mathbf{R}$ *and* $f(0) < f(p)$ *for all* $p \in \mathbf{R}^n \setminus \{0\}$. *If* u *satisfies* (2.8), *then for* $\mathscr{L}^n$*-almost every* $x \in \tilde{\Omega}$

$$\operatorname{ess\,inf}_{\tilde{\Omega}} u_0 \leq u(x) \leq \operatorname{ess\,sup}_{\tilde{\Omega}} u_0$$

Remark 2.8. We do not require that f be strictly convex or satisfy any other structural assumption.

Proof. Let $c_0 = \operatorname{ess\,inf}_{\tilde{\Omega}} u_0$ and $c^0 = \operatorname{ess\,sup}_{\tilde{\Omega}} u_0$. When $m = 1$, $P_z = 0$ for all $z \in \mathbf{S}^{m-1}$. Furthermore,

$$u_0(\tilde{\Omega}) \subset K_{c_0,-1} \cap K_{c^0,1}$$

and $f(pP_z) = f(0) < f(p)\mathbf{R}^n \setminus \{0\}$. Applying proposition 2.5 twice, once for $v = -1$ and $q = c_0$ and once for for $v = 1$ and $q = c^0$ we find that $u(x) \in K_{c_0,R,-1} \cap K_{c^0,R,1}$ for $\mathscr{L}^n$-almost every $x \in \tilde{\Omega}$. $\square$

Corollary 2.9. *Suppose* f *is strictly increasing in* $|Q_i^T|$ *for each* $i = 1, \ldots, n$. *If* K *is the convex hull of* $u_0(\tilde{\Omega})$ *and if* u *satisfies* (2.8), *then for* $\mathscr{L}^n$*-almost every* $x \in \mathbf{R}^n$ $u(x) \in K$.

Proof. Let X be a countable set of couples $(z, q) \in \mathbf{S}^{m-1} \times \mathbf{R}^m$ for which

$$K = \bigcap_{(z,q)\in X} \{x : (x - q) \cdot z \leq 0\}.$$

If X is empty, then $K = \mathbf{R}^m$ and we are done. Otherwise, let $(z, q) \in X$.

$$u_0(\tilde{\Omega}) \subset K_{q,z}.$$

Furthmore, P_z strictly decreases the Euclidean distance in $\mathbf{R}^m$ and hence for $Q \in \mathrm{R}^{n\times m}$

$$f(QP_z) = f(Q_1^T P_z, \ldots, Q_n^T P_z) < f(Q_1^T, \ldots, Q_n^T) = f(Q)$$

unless $Q_i^T P_z = Q_i^T$ for all $i = 1, \ldots, n$, in which case $Q = 0$. By proposition 2.5, $u(x) \in K_{q,z}$ for $\mathscr{L}^n$-almost every $x \in \tilde{\Omega}$. Since $(v, z) \in X$ was arbitrary and chosen from a countable set, we also have

$$u(x) \in \bigcap_{(z,q)\in X} K_{q,z} = K$$

for $\mathscr{L}^n$-almost every $x \in \tilde{\Omega}$. $\square$

3 Problems Without the Convex Hull Property

In this section, we discuss variational problems (proposition 3.2 and proposition 3.3) which do not posses the convex hull property. We give convex, linear growth functions for which solutions of (2.8) have two dimensional range where the data is one dimensional.

The following simple lemma will be used in proposition 3.2 to force an elliptic function to be convex and have linear growth. Let $\tilde{f}$ be a real valued function defined on $\mathbf{R}^{n\times m}$ and assume that $\tilde{f}$ is convex on the set $\Sigma_L := \{Q : \tilde{f}(Q) \leq L\}$ and Σ_L is bounded for some $L > 0$. Then $\mathrm{Lip}_{\Sigma_L}(\tilde{f}) < \infty$. Define a function f on $\mathbf{R}^{n\times m}$ by

$$f(Q) = \max\{\tilde{f}(Q), \mathrm{Lip}_{\Sigma_L}(\tilde{f})\mathrm{dist}(Q, \Sigma_L) + L\}. \tag{3.1}$$

First an elementary

Lemma 3.1. *The function f defined in* (3.1) *is convex and has linear growth.*

Proof. The linear growth of f follows from the linear growth of the distance function and the fact that Σ_L is bounded. Let $P, Q \in \mathbf{R}^{N\times n}$, $\lambda \in [0, 1]$, $R_\lambda = \lambda P + (1-\lambda)Q$ and P_λ, Q_λ be nearest points in Σ_L to P and Q respectively. Suppose that $P, Q \in \mathbf{R}^{N\times n} \setminus \Sigma_L$ and $R_\lambda \in \Sigma_L$. Then

$$f(R_\lambda) \leq \lambda f(P_\lambda) + (1-\lambda)f(Q_\lambda) \leq \lambda f(P) + (1-\lambda)f(Q)$$

where the first inequality follows from the convexity of Σ_L and $\tilde{f}$ while the second follows from the definition of P_λ and Q_λ. Suppose that $P \in \mathbf{R}^{N\times n} \setminus \Sigma_L$ and $Q, R_\lambda \in \Sigma_L$. Then

$$f(R_\lambda) \leq \lambda f(P_\lambda) + (1-\lambda)f(Q) \leq \lambda f(P) + (1-\lambda)f(Q)$$

follows similarly. Finally, suppose $P, R_\lambda \in \mathbf{R}^{N\times n} \setminus \Sigma_L$ and $Q \in \Sigma_L$. Then

$$f(R_\lambda) - \mathrm{Lip}_{\Sigma_L}(\tilde{f})|R_\lambda - Q| \leq f(Q)$$

combined with

$$f(R_\lambda) \leq \lambda f(P) + (1-\lambda)(f(R_\lambda) - \mathrm{Lip}_{\Sigma_L}(\tilde{f})|R_\lambda - Q|)$$

gives the desired inequality. The remaining cases follow easily from the fact that $\tilde{f}$ and $\mathrm{dist}(\cdot, \Sigma_L)$ are convex functions. □

In the following, $\Omega \subset \mathbf{R}^2$ is the unit disk centered at the origin. We will only consider $u \in \mathscr{M}$ for which Du is absolutely continuous and use the variables

$$\left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_2}\right) \leftrightarrow (Q_{11}, Q_{12}, Q_{21}, Q_{22}), \quad Q = \left(\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right).$$

If f is differentiable at Q, then $D^2 f = \{f_{Q_{ij}Q_{kl}}\}_{i,j,k,l=1}^2$ will denote the Hessian.

The first example allows us to explicitly compute solutions of (2.8).

Proposition 3.2. *Let f be as in* (3.1) *where*

$$\tilde{f}(Du) = |Du_1|^2 + |Du_2|^2 + \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1}. \tag{3.2}$$

There exist u_0 so that if u is a solution of (2.8)*,*

$$\mathscr{L}^2\{x \in \tilde{\Omega} : u(x) \notin (H_{\text{convex}} u_0(\tilde{\Omega}))\} > 0.$$

Proof. To establishes the convexity and linear growth of f, note that $D^2\tilde{f}$ is symmetric and strictly positive. Let

$$u_0(x) = e_1(2x_1^2 - 1), \quad x \in \mathbf{R}^2. \tag{3.3}$$

Clearly $u_0 \in W^{1,1}_{\text{loc}}(\mathbf{R}^2; \mathbf{R}^2)$ and the convex hull K of $u_0(\Omega)$ is the one dimensional set $[-1, 1] \times \{0\} \subset \mathbf{R}^2$.

Suppose to the contrary that $u(\tilde{\Omega})$ is contained in K. Then there is a function $\mu \in \mathrm{BV}(\tilde{\Omega})$ for which $u(x) = \mu(x)e_1$ for a.e. $x \in \tilde{\Omega}$ and $\mu(x) = \mu_0(x)$ for a.e. $x \in \tilde{\Omega} \setminus \Omega$ where we define $\mu_0(x) = 2x_1^2 - 1$. Define a CLG function

$$\Phi(p) \equiv f(Q_p) = |p|^2, \quad Q_p = \left(\begin{array}{cc} p_1 & 0 \\ p_2 & 0 \end{array} \right).$$

Clearly μ is a solution to

$$\mu \in \mathscr{N}, \quad J[\mu] \equiv \int_{\tilde{\Omega}} \Phi(D\mu)\, dx \le J[\nu],$$
$$\forall \nu \in \mathscr{N} \equiv \{\nu \in \mathrm{BV}(\tilde{\Omega}) : \nu(x) = \mu_0(x) \text{ for a.e. } x \in \tilde{\Omega} \setminus \Omega\}.$$

By modifying the argument around (2.9), one may show that there is $0 < M < \infty$ independent of L so that

$$\operatorname{ess\,sup}_{x\in\Omega} |D\mu| < M.$$

Choosing L sufficient large in (3.1), we find that $\Phi(D\mu) = |D\mu|^2$. Consequently μ minimizes the Dirichlet integral and is harmonic. In fact

$$\mu(x) = x_1^2 - x_2^2 \tag{3.4}$$

Now consider a smooth solution $\tilde{u}_t := (\mu_t, \nu_t)$ solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial \mu_t}{\partial t} = \Delta\mu_t + \dfrac{\partial^2 \nu_t}{\partial x_1^2}, \quad \mu_0 = \mu, \\ \dfrac{\partial \nu_t}{\partial t} = \Delta\nu_t + \dfrac{\partial^2 \mu_t}{\partial x_1^2}, \quad \nu_0 = 0 \\ \tilde{u}_t(x) = u_0(x), 0 \le t < \infty, x \in \partial\Omega \end{cases} \tag{3.5}$$

with $D\tilde{u}_t \in \Sigma_L$, at least for small time. One has

$$\begin{aligned} \frac{d}{dt}\int_{\tilde{\Omega}} f(D\tilde{u}_t)\, dx &= \sum_{i,j=1}^{2} \int_\Omega f_{Q_{ij}}(D\tilde{u}_t) \left(\frac{\partial^2 (\tilde{u}_t)_i}{\partial x_j \partial t} \right) dx \\ &= -\sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial}{\partial x_j} f_{Q_{ij}}(D\tilde{u}_t) \right)^2 dx \equiv \eta(t). \end{aligned}$$

However, according (3.4) and (3.5),

$$\eta(0) = -\sum_{i,j=1}^{2} \int_\Omega \left(f_{Q_{ij}Q_{kl}}(D\tilde{u}_0) \frac{\partial^2 (u_0)_k}{\partial x_l \partial x_j} \right)^2 dx = -\int_\Omega \left(\frac{\partial^2 \mu}{\partial x_1^2} \right)^2 dx < 0.$$

This contradicts the minimality of $I[u]$ since $\tilde{u}_t \in \mathscr{M}$ and

$$I[\tilde{u}_t] = \eta(t) < \eta(0) = I[u]$$

for sufficiently small, positive t. $\square$

Motivated by the above propositions, we give an example of an immersion of the disk into $\mathbf{R}^3$ which is not contained in the convex hull of its boundary data, an embedded circle.

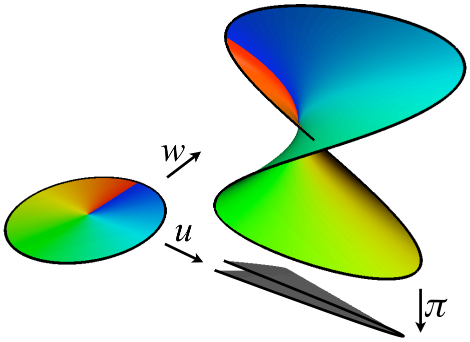

Figure 1: A resolution of the image of u in proposition 3.2. Note that the image lies outside the convex hull of the data (the black curves).

Let Ω be as above. Define an embedding of the unit circle $\partial\Omega$ by $x \in \mathbf{S}^1 \mapsto u_0(x) \in \mathbf{R}^3$ where

$$u_0(x) = e_1(2x_1^2 - 1) + e_2\epsilon x_2 + e_3(x_1 + x_2) \tag{3.6}$$

The above analysis shows that for sufficiently large L in (3.1), if w satisfies (2.8) with f defined by (3.2) and u_0 given by (3.6), then w is unique and is the equilibrium solution of (3.5). Solving we find

$$w(x) = \frac{e_1}{3}(4x_1^2 - 2x_2^2 - 1) + e_2\left(\epsilon x_2 - \frac{2}{3}(x_1^2 + x_2^2 - 1)\right) + e_3(x_1 + x_2), \quad x \in \Omega. \tag{3.7}$$

The image of the disk Ω under w can be seen in figure 1. The solution given in proposition 3.2 can resolved this way by setting $\epsilon = 0$ and projecting by π onto the x_1, x_2-axis.

One may wonder whether the convex hull property is lost due to the ellipticity of (3.2). The following proposition shows that the convex hull property may not hold for honest convex functions of linear growth.

Proposition 3.3. *Let*

$$f(Du) = \left(|Du_1|^2 + |Du_2|^2 + \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1}\right)^{1/2}. \tag{3.8}$$

There exist u_0 *so that if* u *is a solution of* (2.8),

$$\mathscr{L}^2\{x \in \tilde{\Omega} : u(x) \notin H_{\text{convex}}(u_0(\tilde{\Omega}))\} > 0$$

Proof. To see that f is convex, make the change of variables

$$(Q_{11}, Q_{12}, Q_{21}, Q_{22}) \mapsto \left(\frac{Q_{11}+Q_{12}}{\sqrt{2}}, \frac{Q_{11}-Q_{12}}{\sqrt{2}}, Q_{21}, Q_{22}\right) \equiv (\zeta, \eta, \xi, \chi).$$

In these coordinates (a rotation of Q) (3.8) becomes $(3\zeta^2/2+\eta^2/2+\xi^2+\chi^2)^{1/2}$. This function is convex by the triangle inequality. The linear growth of f is clear from its definition.

Let

$$u_0(x) = e_1(x_1^2 - 1), \quad x \in \tilde{\Omega}. \tag{3.9}$$

Again, $u_0 \in W^{1,1}(\Omega; \mathbf{R}^2)$ and the convex hull K of $u_0(\Omega)$ is the one dimensional set $\{te_1 : t \in [-1, 1]\} \subset \mathbf{R}^2$.

Suppose to the contrary that $u(x)$ is contained in K for $\mathscr{L}^2$-almost every $x \in \tilde{\Omega}$. Then there is a function $\mu \in \mathrm{BV}(\tilde{\Omega})$ for which $u(x) = \mu(x)e_1$ for a.e. $x \in \tilde{\Omega}$ and $\mu(x) = \mu_0(x)$ for a.e. $x \in \tilde{\Omega} \setminus \Omega$ where we define $\mu_0(x) = x_1^2 - 1$. In particular, $Du_1 = D\mu, Du_2 = 0$ and thus the variational integral reduces to

$$I[u] = \int_{\tilde{\Omega}} \left(|Du_1|^2\right)^{1/2} dx = \int_{\Omega} |D\mu|\, dx.$$

Clearly μ is a solution to

$$\mu \in \mathscr{N}, \quad J[\mu] \equiv \int_{\tilde{\Omega}} |D\mu|\, dx \le J[\nu],$$
$$\forall \nu \in \mathscr{N} \equiv \{\nu \in \mathrm{BV}(\tilde{\Omega}) : \nu(x) = \mu_0(x) \text{ for a.e. } x \in \tilde{\Omega} \setminus \Omega\}.$$

Unlike in the proof of proposition 3.3, it is difficult to work with a parabolic equation associated with (3.8). Instead, we elect to work with $I[\cdot]$ directly. We use the definitions for

$$\Omega = \mathscr{R} \cup \Omega_1 \cup \Omega_2, \quad \partial\mathscr{R} = \partial_1\mathscr{R} \cup \partial_2\mathscr{R}$$

found in theorem 4.4 in the appendix when $t = \infty$. Note that μ agrees with (4.7) a.e. in Ω for $t = \infty$. Let $\nu \subset C_0^\infty(\Omega)$ and for $\epsilon > 0$ and define $u_\epsilon = \mu e_1 + \epsilon\nu e_2$. Consider

$$\begin{aligned} I[u_\epsilon] &= \int_{\tilde{\Omega}} \left(|D\mu|^2 + \epsilon^2|D\nu|^2 + \epsilon\frac{\partial\mu}{\partial x_1}\frac{\partial\nu}{\partial x_1}\right)^{1/2} dx \\ &= \int_{\Omega_1} \left(|D\mu|^2 + \epsilon^2|D\nu|^2 + \epsilon\frac{\partial\mu}{\partial x_1}\frac{\partial\nu}{\partial x_1}\right)^{1/2} dx + \int_{\Omega_2} \left(|D\mu|^2 + \epsilon^2|D\nu|^2\right)^{1/2} dx + \int_R \epsilon|D\nu|\, dx. \end{aligned}$$

Differentiating the above expression with respect to ϵ and then setting ϵ equal to zero we find

$$\begin{aligned} \frac{d}{d\epsilon} I[u_\epsilon]\big|_{\epsilon=0} &= \int_{\Omega_1} \frac{1}{2}|D\mu|^{-1}\frac{\partial\mu}{\partial x_1}\frac{\partial\nu}{\partial x_1} + \int_R |D\nu|\,dx = \int_{\Omega_1}\frac{\operatorname{sgn}(x_1)}{2}\frac{\partial\nu}{\partial x_1} + \int_R |D\nu|\,dx \\ &= -\frac{1}{2}\int_{\partial R_1}\nu\,d\mathscr{H}^1 + \int_R |D\nu|\,dx. \end{aligned}$$

By approximation, we may choose a sequence $\{\nu_i\}_{i=1}^\infty$ in $C_0^\infty(\Omega)$ with $\nu_i \to \chi_R \in \mathrm{BV}(\Omega)$ with

$$\nu_i \to \nu \text{ in } L^1(\partial R) \text{ and } D\nu_i \to D\nu \text{ in } L^1(\Omega).$$

Thus,

$$\begin{aligned} \lim_{i\to\infty}\frac{d}{d\epsilon}I[u_\epsilon]\big|_{\epsilon=0} &= \lim_{i\to\infty} -\frac{1}{2}\int_{\partial R_1}\nu_i\,d\mathscr{H}^1 + \int_R |D\nu_i|\,dx \\ &= -\frac{1}{2}\mathscr{H}^1(\partial R_1) + 0\mathscr{H}^2(R) < 0. \end{aligned}$$

This contradicts the minimality of u and the conclusion follows. □

4 Appendix

Consider the following variational problem for scalar valued functions of bounded variation;

$$\begin{gathered} u \in \mathscr{N}, \quad J[u] \equiv \int_{\tilde\Omega} |Du|\,dx \le J[v], \\ \forall u \in \mathscr{N} \equiv \{v \in \mathrm{BV}(\tilde\Omega) : v(x) = u_0(x) \text{ for a.e. } x \in \tilde\Omega \setminus \Omega\}. \end{gathered} \tag{4.1}$$

There is a parabolic equation associated with (4.1) called the coarea flow or total variation flow. The coarea flow is the L^2 gradient descent of a function of bounded variation with respect to the total variation $J[\cdot]$. In classical form, the Cauchy problem for the coarea flow reads

$$\frac{\partial\mu_t}{\partial t} = \operatorname{div}\left(\frac{D\mu_t}{|D\mu_t|}\right) \tag{4.2}$$

$$\mu_0(x) = h(x), \quad x \in \Omega, \tag{4.3}$$

$$\mu(x) = g(x), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{4.4}$$

One cannot in general expect a classical solution to (4.2-4.4) to exist. Instead, a unique generalized solution ([HZ94]) $\mu_t \in L^\infty([0,\infty);\mathrm{BV}(\Omega))$ with $\mu_t' \in L^2(\Omega \times [0,\infty))$ is known to exist and satisfies

$$\int_0^s \int_\Omega \mu_t'(\nu - \mu_t) + |D\nu| - |D\mu_t| \, dx + \int_{\partial\Omega} |\nu - g| \, d\mathscr{H}^{n-1} \, dt \geq 0 \tag{4.5}$$

for all $\nu \in \mathrm{BV}(\Omega)$ and $s \geq 0$. If Ω is convex and h and g satisfy a Lipschitz condition, then $\mu_t \in L^\infty([0,\infty);\mathrm{Lip}(\Omega))$. We will prove the following

Theorem 4.1. *There exist real analytic data h and g and a bounded real analytic domain Ω for which the unique, generalized solution to (4.2-4.4) is uniformly Lipschitz but never differentiable and $\mu_t' \notin L^\infty(\Omega \times (0,\epsilon))$ for any $\epsilon > 0$. The solution μ_t converges uniformly to a solution $\mu = \mu_\infty$ of* (4.1) *with an exponential rate.*

Remark 4.2. A feature of the coarea flow is that it seems to be a tool for finding area minimizing hypersurfaces. As was pointed out in [HZ94] example 5.2, it is interesting to note that the levels of h are stationary hypersurfaces but not locally area minimizing. In contrast, a solution to the level set equation ([ES91]) with data h and g is then constant in time.

Lemma 4.3. *There is a unique solution $a : \mathbf{R}_+ \to \mathbf{R}$ to the ordinary differential equation*

$$a'(t) = \frac{1}{4a^2(t)} - \frac{1}{4a(t)\sqrt{1-a(t)^2}}, \quad \lim_{t\downarrow 0} a(t) = 0. \tag{4.6}$$

The solution $a(t)$ is monotonically increasing and the estimate

$$c_1 e^{-d_1 t} \leq \left| a(t) - \frac{\sqrt{2}}{2} \right| \leq c_2 e^{-d_2 t}$$

holds for some positive constants c_1, c_2, d_1, d_2.

Theorem 4.1 is now a consequence of

Theorem 4.4. *Let Ω be the two dimensional unit disk centered at the origin. Define Cauchy data*

$$h(x) = g(x) = \mu_0(x) = 2x_1^2 - 1 = 1 - 2x_2^2, \quad x \in \mathbf{R}^2.$$

Let $a : \mathbf{R}_+ \to \mathbf{R}$ *be as in lemma 4.3. For* $t \geq 0$ *let* $z(t) = a(t)^2$ *and* $b(t) = \sqrt{1 - a(t)^2}$ *and* $\mathscr{R}(t) \subset \Omega$ *be the rectangle parallel with the coordinate axes and horizontal and vertical side lengths* $2a(t)$ *and* $2b(t)$ *respectively and centered at the origin. Let* $\partial_1\mathscr{R}(t)$ *be the vertical part and* $\partial_2\mathscr{R}(t)$ *the horizontal part of the boundary of* $\mathscr{R}(t)$*. Define*

$$u(x,t) = \begin{cases} x_1^2 & |x_1| \geq a(t) \equiv \Omega_1(t), \\ 1 - x_2^2 & |x_2| \geq b(t) \equiv \Omega_1(t), \\ z(t) & x \in \mathscr{R}(t). \end{cases} \tag{4.7}$$

Then $\mu_t \in L^\infty([0,\infty); \mathrm{Lip}(\Omega))$ *with* $\mu_t' \in L^2(\Omega \times [0,\infty))$*and* μ_t *satisfies* (4.5)*.*

We describe briefly how one arrives at (4.7). We first assume that u is defined by (4.7) for an unknown function $z : \mathbf{R}_+ \to \mathbf{R}$ and then use test functions v in (4.5) to find an ordinary differential equation satisfied by z. Let $\eta \in C_0^\infty(\Omega)$ and choose $v = u + \epsilon\eta$ for $\epsilon > 0$. In the sequel, the dependence on t will be omitted. Then the integrand of (4.5) becomes

$$\epsilon \int_{\mathscr{R}} z'\eta + |D\eta|\, dx + \int_{\Omega\setminus\mathscr{R}} |Du + \epsilon D\eta| - |Du|\, dx \geq 0.$$

Since $|Du| > 0$ on $\Omega \setminus \mathscr{R}$, dividing by ϵ and bounded convergence implies

$$\int_{\mathscr{R}} z'\eta + |D\eta|\, dx + \int_{\Omega\setminus\mathscr{R}} \frac{Du}{|Du|} \cdot D\eta\, dx \geq 0. \tag{4.8}$$

Since u is constant in the x_1 or x_2 variable in the complement of $\mathscr{R}$, we have

$$\begin{aligned} \int_{\Omega\setminus\mathscr{R}} \frac{Du}{|Du|} \cdot D\eta\, dx &= \int_{\mathscr{R}\cap\{x_1\geq a\}} \frac{\partial\eta}{\partial x_1}\, dx - \int_{\mathscr{R}\cap\{x_1\leq a\}} \frac{\partial\eta}{\partial x_1}\, dx \\ &\quad - \int_{\mathscr{R}\cap\{x_2\geq b\}} \frac{\partial\eta}{\partial x_2}\, dx + \int_{\mathscr{R}\cap\{x_2\leq b\}} \frac{\partial\eta}{\partial x_2}\, dx \\ &= \int_{\partial\mathscr{R}} \sigma\eta\, d\mathscr{H}^1, \quad (\eta \text{ has compact support}) \end{aligned}$$

where $\sigma = 1$ on the horizontal part of the boundary of $\mathscr{R}$ and $\sigma = -1$ on the vertical part of the boundary of $\mathscr{R}$. Choosing a sequence $\eta_i|\mathscr{R} \to \chi_{\mathscr{R}}$, this and (4.8) implies

$$z'\mu(\mathscr{R}) + 2(a - b) \geq 0. \tag{4.9}$$

On the other hand, choosing $v = u - \epsilon\eta$ for $\epsilon > 0$ implies

$$-\epsilon \int_{\mathscr{R}} z'\eta + |D\eta|\,dx + \int_{\Omega\setminus\mathscr{R}} |Du - \epsilon D\eta| - |Du|\,dx \geq 0.$$

Dividing by ϵ and sending ϵ to zero implies the contrary inequality;

$$\int_{\mathscr{R}} z'\eta + |D\eta|\,dx + \int_{\Omega\setminus\mathscr{R}} \frac{Du}{|Du|} \cdot D\eta\,dx \leq 0. \tag{4.10}$$

Choosing η as before implies then

$$z'\mu(\mathscr{R}) + 2(a - b) \leq 0. \tag{4.11}$$

As $\mu(\mathscr{R}) = 4ab$, (4.9) and (4.11) imply that any solution of the form (4.7) must satisfy

$$z' = \frac{b-a}{2ab}.$$

Changing variables $z(t) = a(t)^2$ gives (4.6).

Now we prove that the function given by (4.7) where z solves (4.6) is in fact a solution of (4.5).

Proof of Theorem 1. Since u is nonincreasing and Ω is convex, it is sufficient to check (4.5) for $v \in BV(\Omega)$ with

$$v \in C^\infty(\Omega), \quad v|\partial\Omega = g, \quad v \geq u. \tag{4.12}$$

Let u be as in (4.7) and $v \in BV(\Omega)$ as in (4.12). Then

$$\begin{aligned}\mathscr{E}(v) &:= \int_\Omega u'(v-u) + |Dv| - |Du|\,dx \\ &= z'\int_{\mathscr{R}} v - z\,dx + \int_\Omega |Dv| - |Du|\,dx\end{aligned}$$

since $u \equiv z$ on $\mathscr{R}$. Applying the change of variables formula gives

$$\mathscr{E}(v) = z'\int_0^\infty \mathscr{L}^2(\{v - z > s\} \cap \mathscr{R}) + \int_0^\infty P_\Omega(\{v > s\}) - P_\Omega(\{u > s\})\,ds$$

Here P_Ω denotes the perimeter relative to Ω; $P_\Omega(E) = \|\partial E\|(\Omega)$. Setting $s + z \mapsto s$ in the first integral gives.

$$\begin{aligned}\mathscr{E}(v) = &\int_z^\infty z'\mathscr{L}^2(\{v > s\} \cap \mathscr{R}) + P_\Omega(\{v > s\}) - P_\Omega(\{u > s\})\,ds \\ &+ \int_0^z P_\Omega(\{v > s\}) - P_\Omega(\{u > s\})\,ds.\end{aligned}$$

By (4.12), $\{u > s\} \subset \{v > s\}$ for all $s \in \mathbf{R}_+$. Then, clearly, $P_\Omega(\{v > s\}) - P_\Omega(\{u > s\}) \geq 0$ ($\partial\{u > s\}$ consists of two parallel line segments) for all $s \in \mathbf{R}_+$ so that in particular, the second integral above is nonnegative. Below we demonstrate that the first integrand is nonnegative for $\mathscr{L}^1$-almost every $s > z$. For fixed s, consider the function

$$\begin{aligned} f(s) &\equiv z'\mathscr{L}^2(\{v > s\} \cap \mathscr{R}) + P_\Omega(\{v > s\}) - P_\Omega(\{u > s\}) \\ &= \frac{b-a}{2ab}\mathscr{L}^2(\{v > s\} \cap \mathscr{R}) + P_\Omega(\{v > s\}) - P_\Omega(\{u > s\}). \end{aligned}$$

By (4.7) and by monotonicity of Lebesgue measure,

$$P_\Omega(\{u > s\}) \leq 2b, \quad \mathscr{L}^2(\{v > s\} \cap \mathscr{R}) \leq \mathscr{L}^2(\mathscr{R}) = 4ab.$$

Thus

$$\begin{aligned} f(s) &\geq 2b - \frac{1}{2b}\mathscr{L}^2(\{v > s\} \cap \mathscr{R}) + P_\Omega(\{v > s\}) - 2b \\ &= P_\Omega(\{v > s\}) - \frac{1}{2b}\mathscr{L}^2(\{v > s\} \cap \mathscr{R}). \end{aligned}$$

By (4.12) and Sard's theorem, for almost every $s \geq z$,

$$\partial\{v > s\} \cap \mathscr{R} = \bigcup_{i=1}^{N} \gamma_i$$

where $\{\gamma\}_{i=1}^{N}$ is a finite, disjoint collection of smooth curves connecting points $(\pm\tilde{a}, \pm\tilde{b})$ where $s \equiv 1 - (\tilde{b})^2 \equiv (\tilde{a})^2$ and

$$P_\Omega(\{v > s\}) = \sum_{i=1}^{N} \text{length}(\gamma_i).$$

Since $s \geq z$, we also have

$$\tilde{a} \geq a, \quad \tilde{b} \leq b.$$

The number if curves N is at least 2. We assume that γ_1 and γ_2 connect $(\pm\tilde{a}, \pm\tilde{b})$ while $\gamma_3, \ldots, \gamma_N$ are the boundaries of disjoint, connected components $G_3, \ldots, G_N$.

Suppose γ_1 connects $(\tilde{a}, \tilde{b})$ with $(-\tilde{a}, \tilde{b})$ and consequently γ_2 connects $(\tilde{a}, -\tilde{b})$ with $(-\tilde{a}, -\tilde{b})$. Then

$$\text{length}(\gamma_1) + \text{length}(\gamma_2) \geq 4\tilde{a}.$$

On the other hand, by the convexity of Ω and definition of g, $\{v > s\} \cap \mathscr{R}$ is contained in the rectangle with corners $(\pm\tilde{a}, \pm b)$ leaving

$$\mathscr{L}^2(\{v > s\} \cap \mathscr{R}) \leq 4\tilde{a}b.$$

Thus

$$f(s) \geq 4\tilde{a} - \frac{1}{2b} 4\tilde{a}b = 2\tilde{a} > 0.$$

In the remaining case, suppose γ_1 and γ_2 connect $(\tilde{a}, \tilde{b})$ with $(\tilde{a}, -\tilde{b})$ and $(-\tilde{a}, \tilde{b})$ with $(-\tilde{a}, -\tilde{b})$ respectively. Let G_1 be the connected component bounded by γ_1 and $\{x_1 = \tilde{a}\}$. Let G_2 be the connected component bounded by γ_1 and $\{x_1 = -\tilde{a}\}$. Since $f(s)$ decreases if the perimeter of $\{v > s\}$ decreases, we may without loss of generality assume that $G_1, \ldots, G_N \subset \mathscr{R}$. Let $\tilde{G}_1$ be the projection of G_1 onto the $\{x_1 = \tilde{a}\}$ axis which preserves the length of the horizontal slices of G_1. Let $\tilde{G}_2$ be the projection of G_2 onto the $\{x_1 = -\tilde{a}\}$ axis which preserves the length of the horizontal slices of G_2. Then $\tilde{G}_i$ is a subgraph (over $x_2 \in (-\tilde{b}, \tilde{b})$) of a continuous, piecewise smooth function g_i with $g_i(-\tilde{b}) = g_i(\tilde{b}) = 0$ and

$$\mathscr{L}^2(\tilde{G}_i) = \mathscr{L}^2(G_i), \quad \text{length}(g_i) \leq \text{length}(\gamma_i), \quad i = 1, 2.$$

Then

$$\begin{aligned} \mathscr{L}^2(G_i) = \mathscr{L}^2(\tilde{G}_i) = \int_{-\tilde{b}}^{\tilde{b}} g_i(x_2)\, dx_2 \leq \int_{-\tilde{b}}^{\tilde{b}} \int_{-\tilde{b}}^{x_2} |g_i'(r)|\, dr\, dx_2 \\ \leq 2\tilde{b} \int_{-\tilde{b}}^{\tilde{b}} \sqrt{1 + (g_i'(r))^2}\, dr = 2\tilde{b}\text{length}(g_i) \leq 2b\text{length}(\gamma_i) \end{aligned} \tag{4.13}$$

for $i = 1, 2$. Since $G_3, \ldots, G_N$ are connected, their projection onto the $\{x_1 = 0\}$ axis are intervals $(c_3, d_3), \ldots, (c_N, d_N) \subset (-\tilde{b}, \tilde{b})$. By making a small perturbation, we may assume that horizontal slices of $G_3, \ldots, G_N$ have vanishing length at the endpoints of the intervals $(c_3, d_3), \ldots, (c_N, d_N)$. Let $\tilde{G}_3, \ldots, \tilde{G}_N$ be the Steiner symmetrization ([EG92]) of $G_3, \ldots, G_N$ with respect to the $\{x_1 = 0\}$ axis and let $g_3, \ldots, g_N$ be the height of $\tilde{G}_3, \ldots, \tilde{G}_N$ defined over (c_3, d_3), $\ldots$, (c_N, d_N). Then

$$\mathscr{L}^2(\tilde{G}_i) = \mathscr{L}^2(G_i), \quad 2\text{length}(g_i) \leq \text{length}(\gamma_i), \quad i = 3, \ldots, N.$$

and

$$
\begin{aligned}
\mathscr{L}^2(G_i) = \mathscr{L}^2(\tilde{G}_i) = 2\int_{c'}^{d'} g_i(x_2)\,dx_2 &\le 2\int_{c'}^{d'}\int_{c'}^{x_2} |g_i'(r)|\,dr\,dx_2 \\
&\le 2(d'-c')\int_{c'}^{d'} \sqrt{1+(g_i'(r))^2}\,dr \\
&= 2(d'-c')\mathrm{length}(g_i) \le 2b\,\mathrm{length}(\gamma_i)
\end{aligned}
\tag{4.14}
$$

In total, (4.13) and (4.14) imply

$$
f(s) = \sum_{i=1}^{N} \left[\mathrm{length}(\gamma_i) - \frac{1}{2b}\mathscr{L}^2(G_i)\right] \ge 0.
$$

This completes the proof. □

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