Math 425 Homework 6

Due Monday, 29. October.

Do all 4 problems. Enjoy!

1. Mimic the proof of Lemma 1, Chapter 5 to prove the following: Let E be a subset of $\mathbb{R}^n$ with finite outer measure[1] and $\mathcal{C}$ a collection of balls of the form

$$B = B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}.$$

Suppose further that for $x \in E$ and $\delta > 0$ there exists $B \in \mathcal{C}$ with $\text{diam} B < \delta$ and $x \in B$. Show that for $\epsilon > 0$ there is a finite, disjoint subcollection $\{B_1, B_2, \ldots, B_J\}$ of $\mathcal{C}$ with

$$m^*(E \setminus \cup_{i=1}^J B_j) < \epsilon.$$

2. Construct a monotone, real valued function which is discontinuous on a dense subset of $[0, 1]$.

3. Which is a function of bounded variation on $[-1, 1]$, $f(x) = x^2 \sin(1/x)$ or $g(x) = x^2 \sin(1/x^2)$ with $f(0) = g(0) = 0$.

4. We say that a function f is Lipschitz if there is an M so that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$. Show that such an f is differentiable almost everywhere. Give an example of a function which is continuous but not Lipschitz on $[0, 1]$.

[1] outer measure in this case means $m^*E = \inf_{E \subset O} m(O)$ where O is open. It should be clear what the measure of balls in $\mathbb{R}^n$ is and that the outer measure is countably subadditive.