Physics of Traffic Flow

In fluid mechanics, the equations of motion are or the form (1D)

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial F(p)}{\partial x}$$

where p is some conserved quantity (such as density, momentum, energy density) and F is a flux function. For example the continuity equation reads as

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\big(\rho(x,t)v(x,t)\big)$$

One of the simplest flows involves a velocity (v) that is a function of the density (ρ), i.e., $v(x,t) = v(\rho)$.

Physics of Traffic Flow -2

For example consider the following functional form

$$v(\rho) = v_m(1 - \frac{\rho}{\rho_m})$$

where $v_m > 0$ is the maximum velocity and $\rho_m > 0$ is the maximum density. This is a crude approximation to car traffic. The maximum velocity is the speed limit and when the maximum density occurs the flow grinds to a halt. The continuity equation then becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left(\rho v_m\left(1 - \frac{\rho}{\rho_m}\right)\right) = -\frac{\partial}{\partial \rho}\left(\rho v_m\left(1 - \frac{\rho}{\rho_m}\right)\right)\frac{\partial \rho}{\partial x}$$

$$= -v_m\left(1 - \frac{2\rho}{\rho_m}\right)\frac{\partial \rho}{\partial x} = -c(\rho)\frac{\partial \rho}{\partial x}$$

Physics of Traffic Flow -3

So we have

$$\frac{\partial \rho}{\partial t} = -c(\rho)\frac{\partial \rho}{\partial x} \qquad c(\rho) = v_m\left(1 - \frac{2\rho}{\rho_m}\right)$$

note:

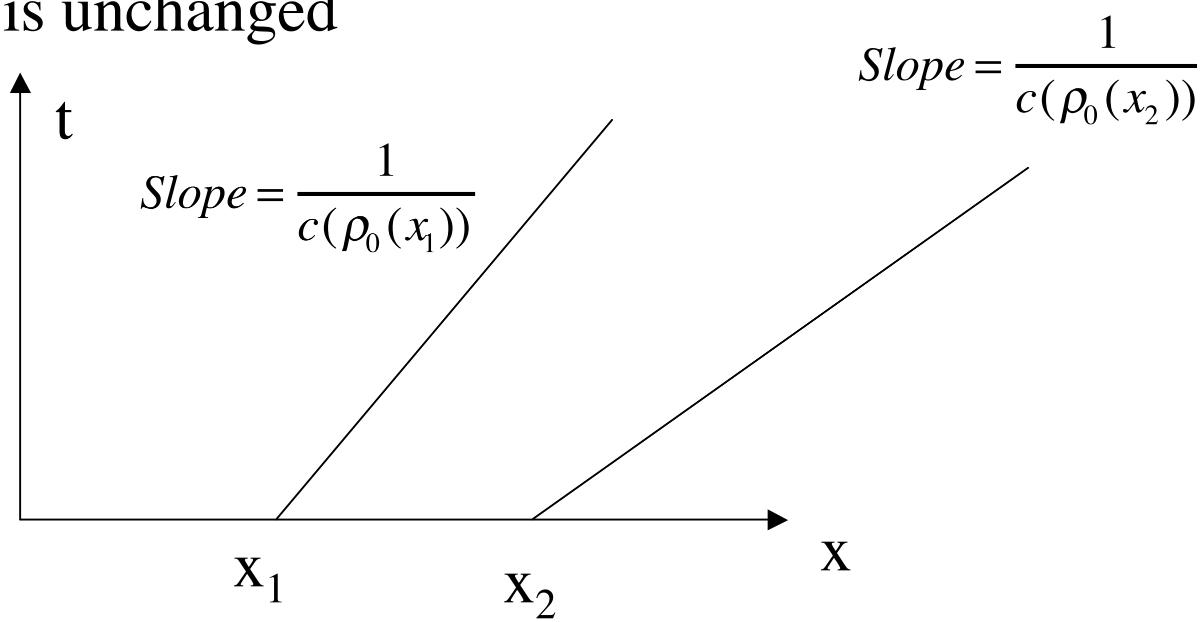
- $c(\rho)$ can be both positive and negative
- $c(\rho)$ is not the speed of the traffic
- $c(\rho)$ is linear in ρ
- $v(\rho) \geq 0$
- $c(\rho) \leq v(\rho)$

Physics of Traffic Flow - Method of Characteristics

It is simple to show that along a characteristic of slope

$$\frac{dt}{dx} = \frac{1}{c(\rho_o(x))}$$

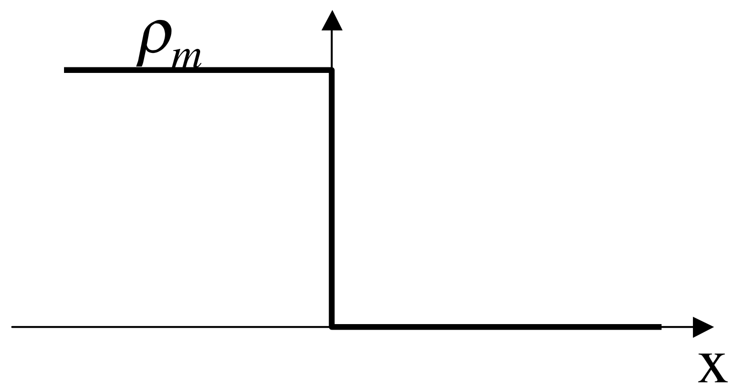
the solution is unchanged

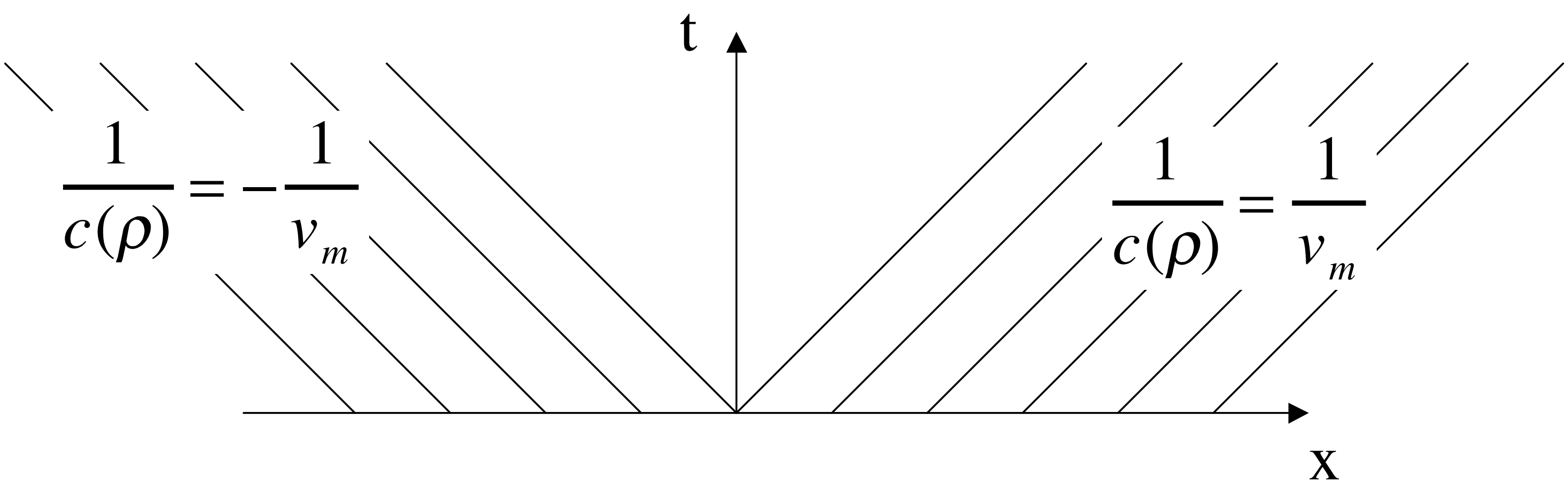


Physics of Traffic Flow - Traffic Light

Consider the initial condition

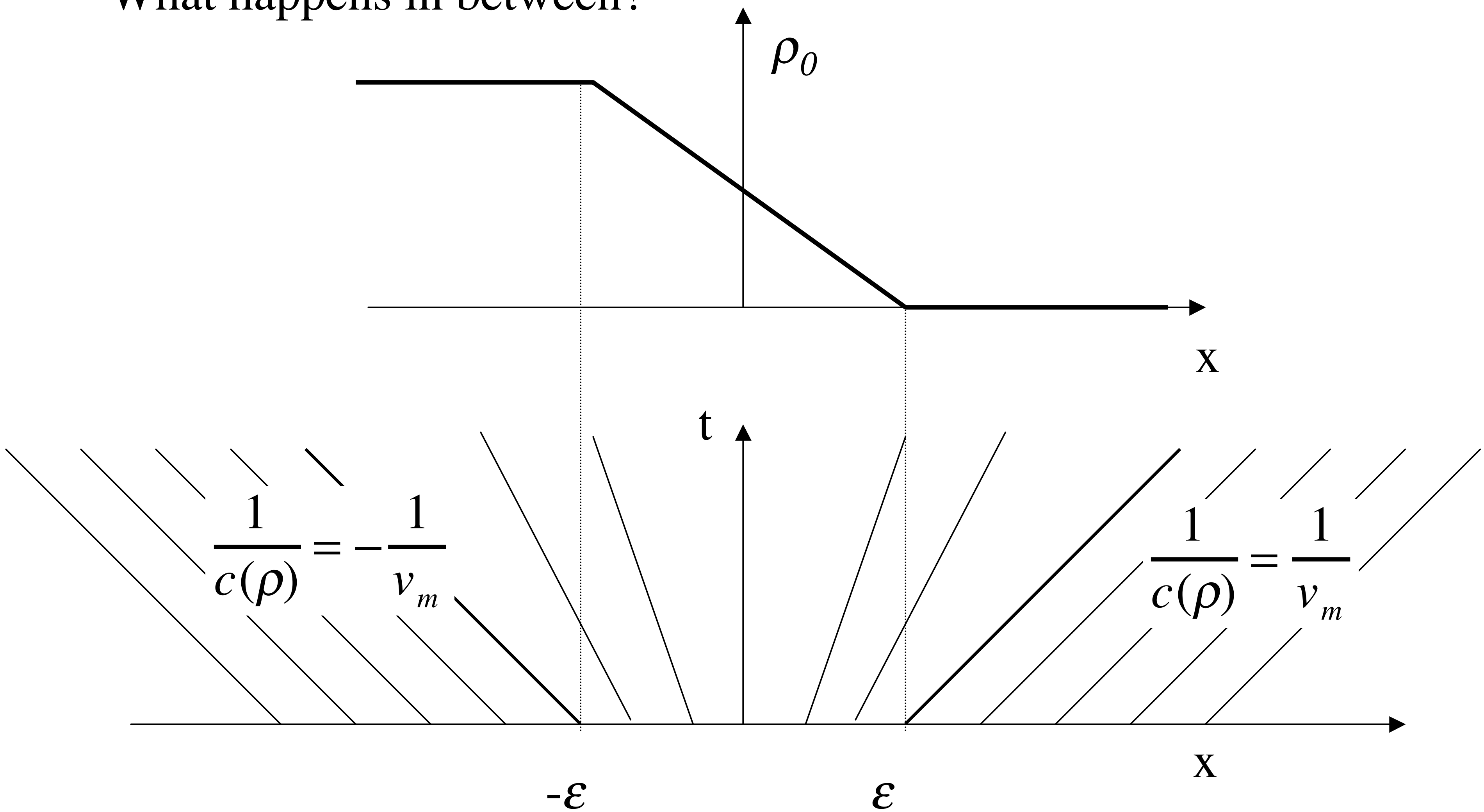
$$\rho(x,0) = \rho_0(x) = \begin{cases} \rho_m & x < 0 \\ 0 & x \geq 0 \end{cases}$$





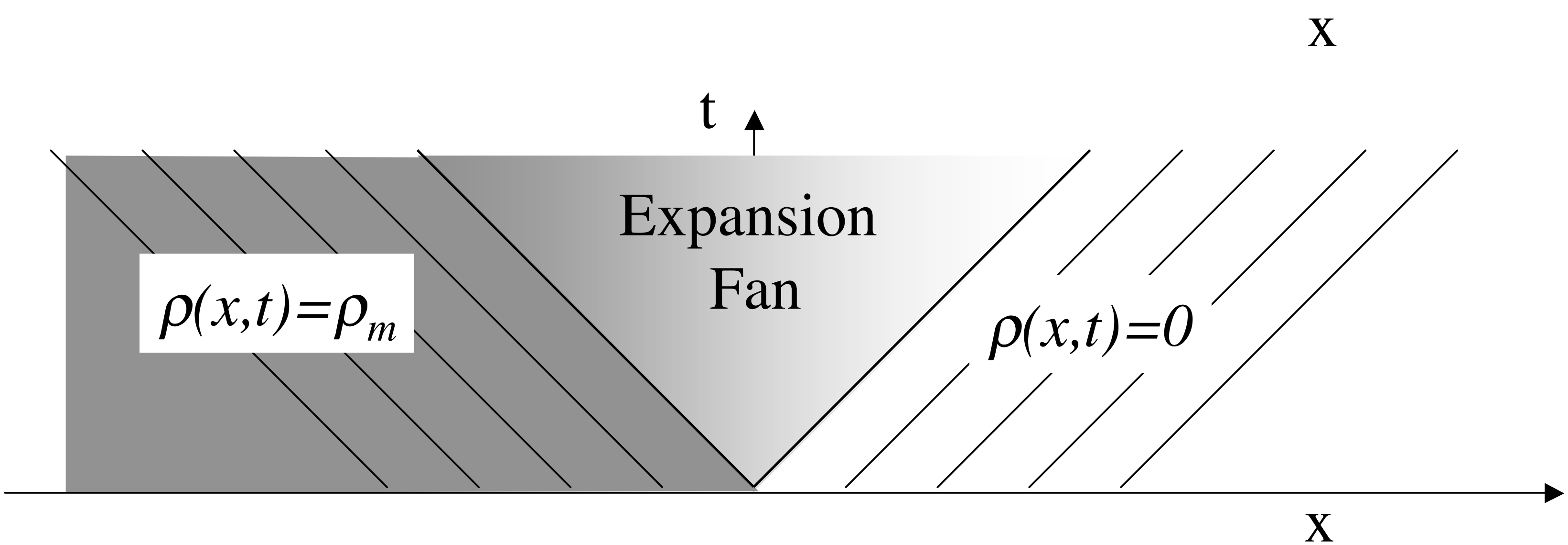
Physics of Traffic Flow - Traffic Light -2

What happens in between?



Physics of Traffic Flow - Traffic Light -3

In the limit as ε →0 then



the solution is then

$$\rho(x,t)=\begin{cases}\rho_m & x\le -v_m t\\ \rho_m\left(1-\dfrac{x}{v_m t}\right) & -v_m t\le x\le v_m t\\ 0 & x\ge v_m t\end{cases}$$

Traffic Program

This program solves the equation $\frac{\partial p(x,t)}{\partial t} = -\frac{\partial F(p)}{\partial x}$ using 3 different methods

- FTCS

$$\rho_i^{n+1} = \rho_i^n - \frac{\tau}{2h}(F_{i+1}^n - F_{i-1}^n)$$

- Lax Scheme

$$\rho_i^{n+1} = \frac{1}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\tau}{2h}(F_{i+1}^n - F_{i-1}^n)$$

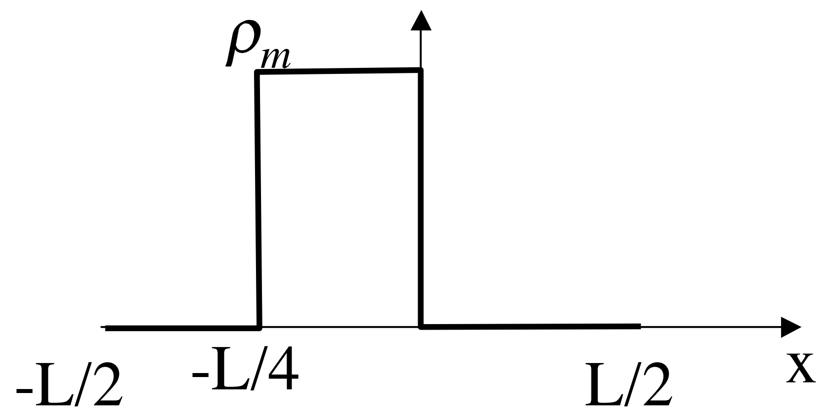
- Lax Wendroff

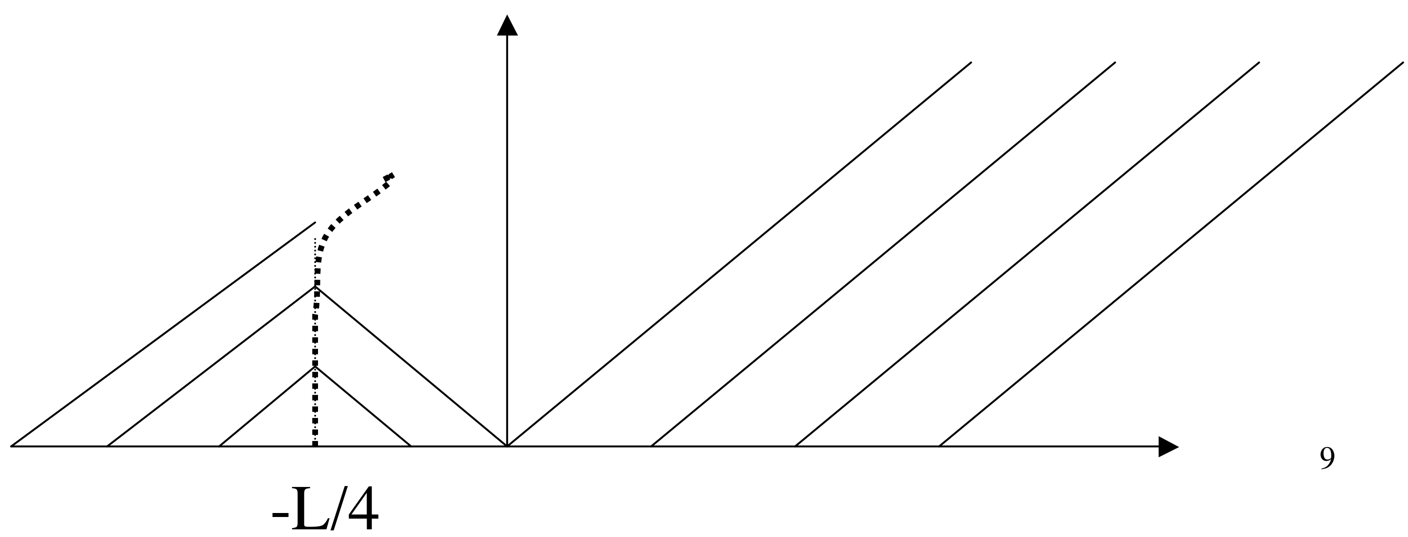
$$\rho_i^{n+1} = \rho_i^n - \frac{\tau}{2h}(F_{i+1}^n - F_{i-1}^n) + \frac{\tau^2}{2h}\left(c_{i+\frac{1}{2}} \frac{F_{i+1}^n - F_i^n}{h} - c_{i-\frac{1}{2}} \frac{F_i^n - F_{i-1}^n}{h} \right)$$

where $c_{i\pm 1/2}^n = c\left(\frac{\rho_{i\pm1}^n + \rho_i^n}{2} \right)$

Traffic Problem - Example

Look at the initial condition $\rho(x,0) = \rho_0(x) = \begin{cases} \rho_m & -L/2 < x < 0 \\ 0 & \text{otherwise} \end{cases}$





Traffic Program

The text has several ways of solving the traffic problem, the simplest is the FTCS, so that

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}F(\rho)$$

Becomes

$$\rho_i^{n+1} = \rho_i^n - \frac{\tau}{2h}\left(F_{i+1}^n - F_{i-1}^n\right)$$

And the LAX method is

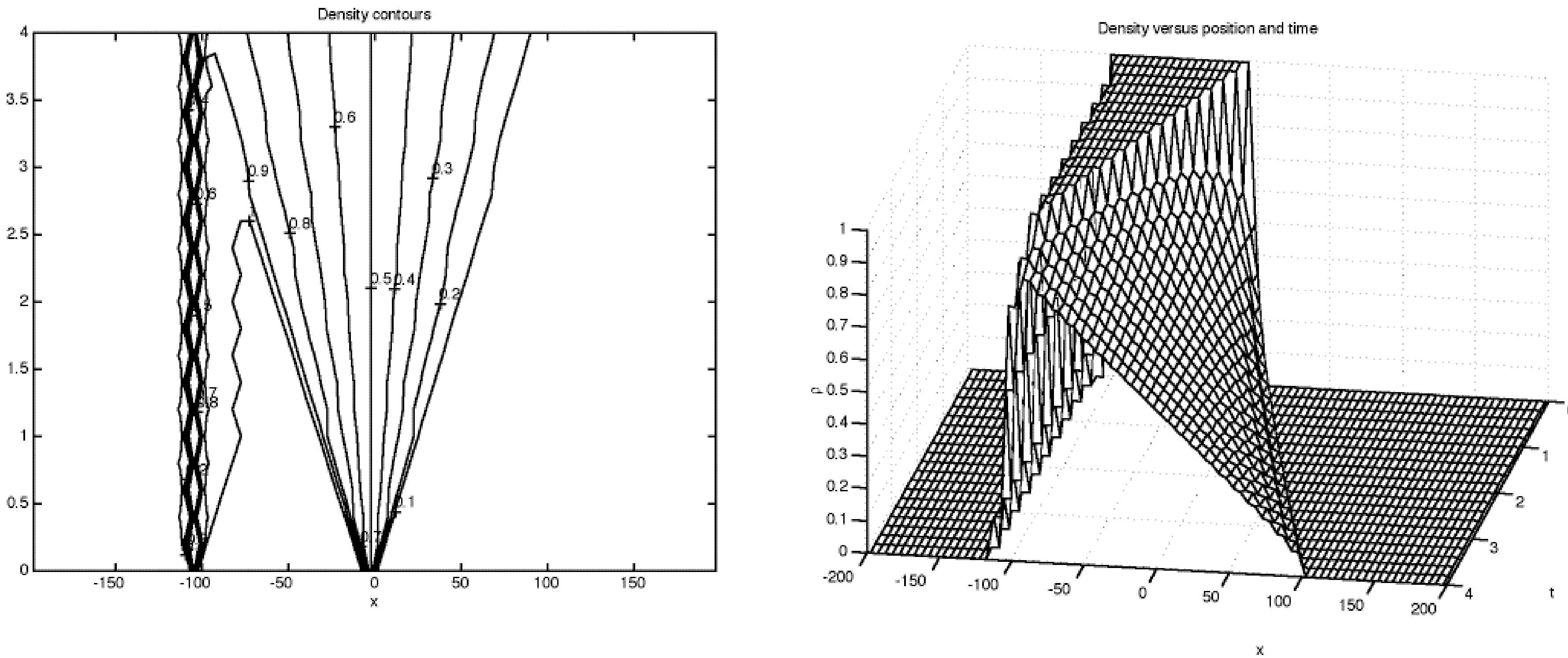
$$\rho_i^{n+1} = \frac{1}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\tau}{2h}\left(F_{i+1}^n - F_{i-1}^n\right)$$

And LAX Wendroff

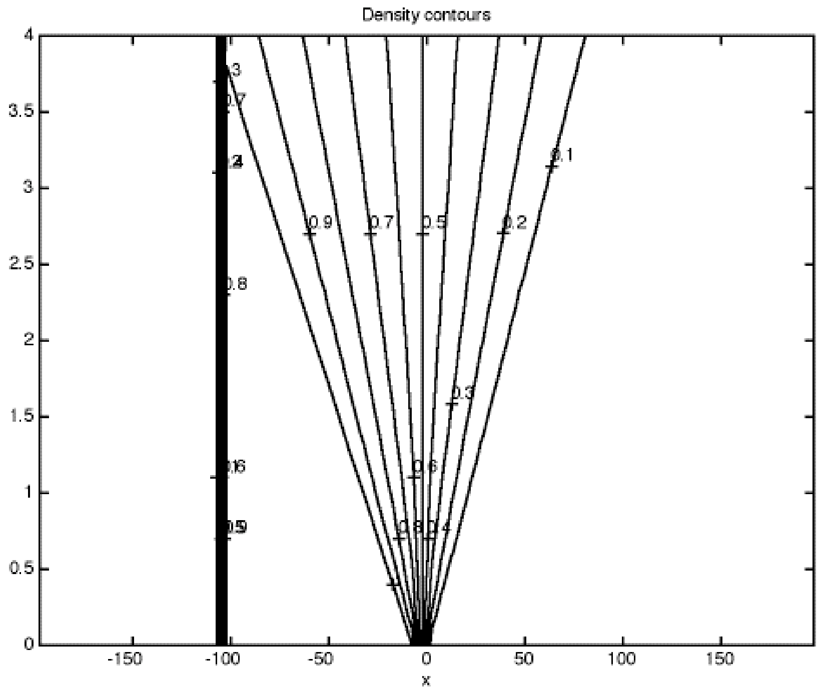
$$\rho_i^{n+1} = \rho_i^n - \frac{\tau}{2h}\left(F_{i+1}^n - F_{i-1}^n\right) + \frac{\tau^2}{2h}\left(c_{i+1/2}\frac{F_{i+1}^n - F_i^n}{h} - c_{i-1/2}\frac{F_i^n - F_{i-1}^n}{h}\right)$$

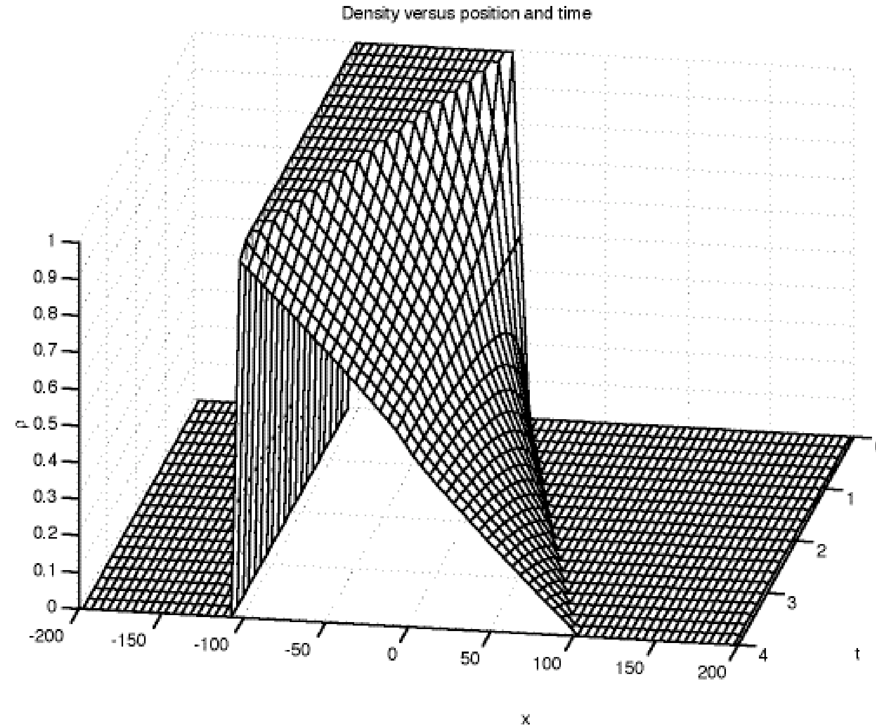
where

$$c_{i\pm 1/2} = c(\rho_{i\pm 1/2}); \quad \rho_{i\pm 1/2} = \frac{1}{2}(\rho_{i\pm 1}^n + \rho_i^n)$$

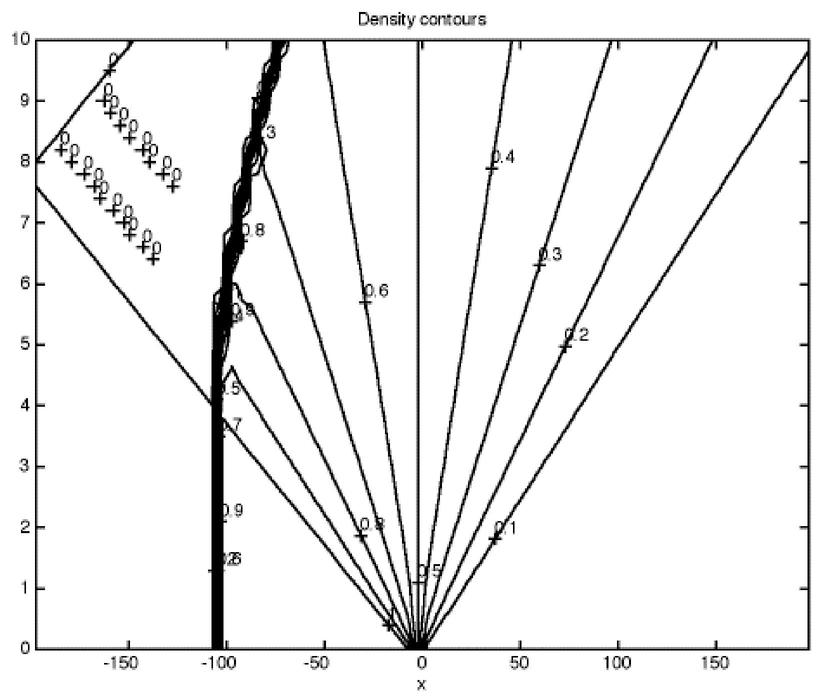


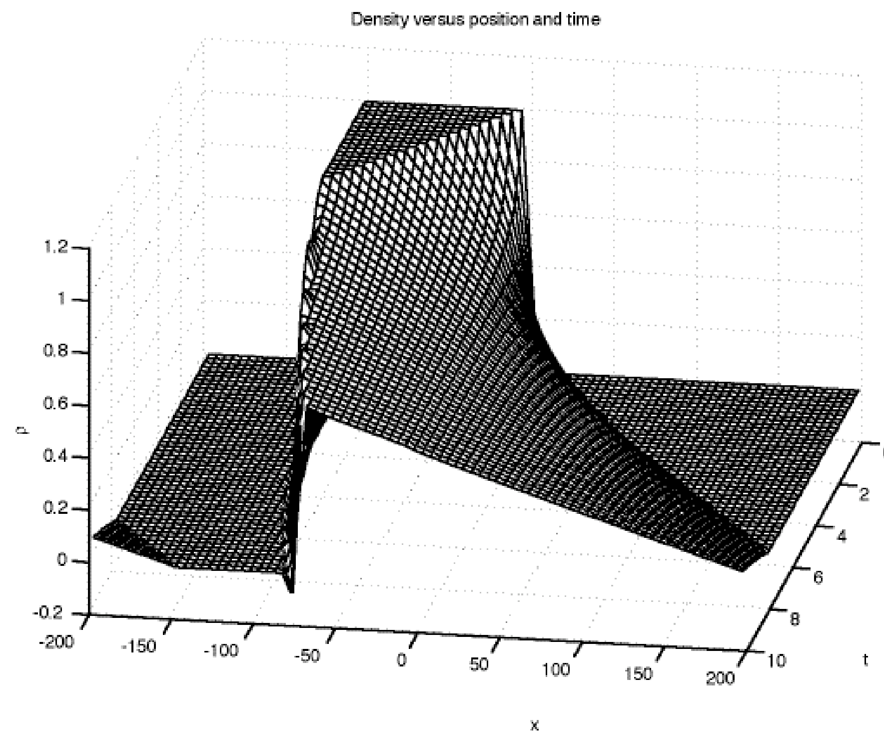
Lax Method τ=0.2, N=80, 20 steps





Lax-Wendroff Method τ=0.2, N=80, 20 steps





Lax-Wendroff Method τ=0.2, N=80, 50 steps