

# The Wave Equation (Chapter 7)

Now let's look at hyperbolic equations, let's start first with the

wave equation 
$$\frac{\partial^2 A(x,t)}{\partial t^2} = c^2 \frac{\partial^2 A(x,t)}{\partial x^2}$$

One standard way to deal with this equation is to break it up into 2 first order equations by defining 2 variables  $P$  and  $Q$

$$P = \frac{\partial A}{\partial t} \quad Q = c \frac{\partial A}{\partial x}$$

The wave equation is then written as

$$\frac{\partial P}{\partial t} = c \frac{\partial Q}{\partial x} \quad \frac{\partial Q}{\partial t} = c \frac{\partial P}{\partial x}$$

or as

$$\frac{\partial \mathbf{a}}{\partial t} = c \mathbf{B} \frac{\partial \mathbf{a}}{\partial x} \quad \text{where} \quad \mathbf{a} = \begin{bmatrix} P \\ Q \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

# The Wave Equation

But lets go back to the original equation

$$\frac{\partial^2 A(x,t)}{\partial t^2} = c^2 \frac{\partial^2 A(x,t)}{\partial x^2}$$

In a finite difference approximation, it can be written as

$$\frac{A_i^{n+1} - 2A_i^n + A_i^{n-1}}{\tau^2} = c^2 \frac{A_{i+1}^n - 2A_i^n + A_{i-1}^n}{h^2}$$

where  $x_i = -\frac{L}{2} + (i-1)h$  and  $t_n = (n-1)\tau$

rewriting, we get

$$A_i^{n+1} = 2A_i^n - A_i^{n-1} + \frac{c^2 \tau^2}{h^2} (A_{i+1}^n - 2A_i^n + A_{i-1}^n)$$

# Wave Equation

One advantage of this approach is that it allows solutions with wave speeds of both signs

$$A_i^{n+1} = 2A_i^n - A_i^{n-1} + \frac{c^2 \tau^2}{h^2} (A_{i+1}^n - 2A_i^n + A_{i-1}^n)$$

whereas the approach based on

$$\frac{\partial \mathbf{a}}{\partial t} = c \mathbf{B} \frac{\partial \mathbf{a}}{\partial x}$$

does not.

One homework problem is to write a program to solve this equation.

# The Advection Equation

We will look now at the simplest hyperbolic equation, the Advection Equation. We shall see that the numerical solution of this simple equation is the cause of endless (numerical) grief. The basic equation is

$$\frac{\partial a(x,t)}{\partial t} = -c \frac{\partial a(x,t)}{\partial x}$$

with an initial condition  $a(x,0)=f_0(x)$  where  $f_0(x)$  is an arbitrary function. The analytical solution to the above equation is easy to obtain as  $a(x,t)=f_0(x-ct)$ . For example, if we have the function

$$a(x,0) = \cos[k(x - x_0)] \exp\left[-\frac{(x - x_0)^2}{2\sigma^2}\right]$$

then

$$a(x,t) = \cos[k(x - ct - x_0)] \exp\left[-\frac{(x - ct - x_0)^2}{2\sigma^2}\right]$$

# FTCS for the Advection Equation

The simplest approach to numerically solving the advection equation is to apply the FTCS as we did for the diffusion equation, so that the equation then becomes

$$\frac{a_i^{n+1} - a_i^n}{\tau} = -c \frac{a_{i+1}^n - a_{i-1}^n}{2h}$$

or

$$a_i^{n+1} = a_i^n - \frac{c\tau}{2h} (a_{i+1}^n - a_{i-1}^n)$$

This method is *unconditionally unstable* for all values of  $\tau$ .

# Von Neumann Stability analysis for FTCS

Assume  $a_j^n = T^n e^{ikjh}$

So that the FTCS method becomes

$$T^{n+1} e^{ikjh} = T^n e^{ikjh} - \frac{c\tau}{2h} T^n (e^{ik(j+1)h} - e^{ik(j-1)h})$$

Or

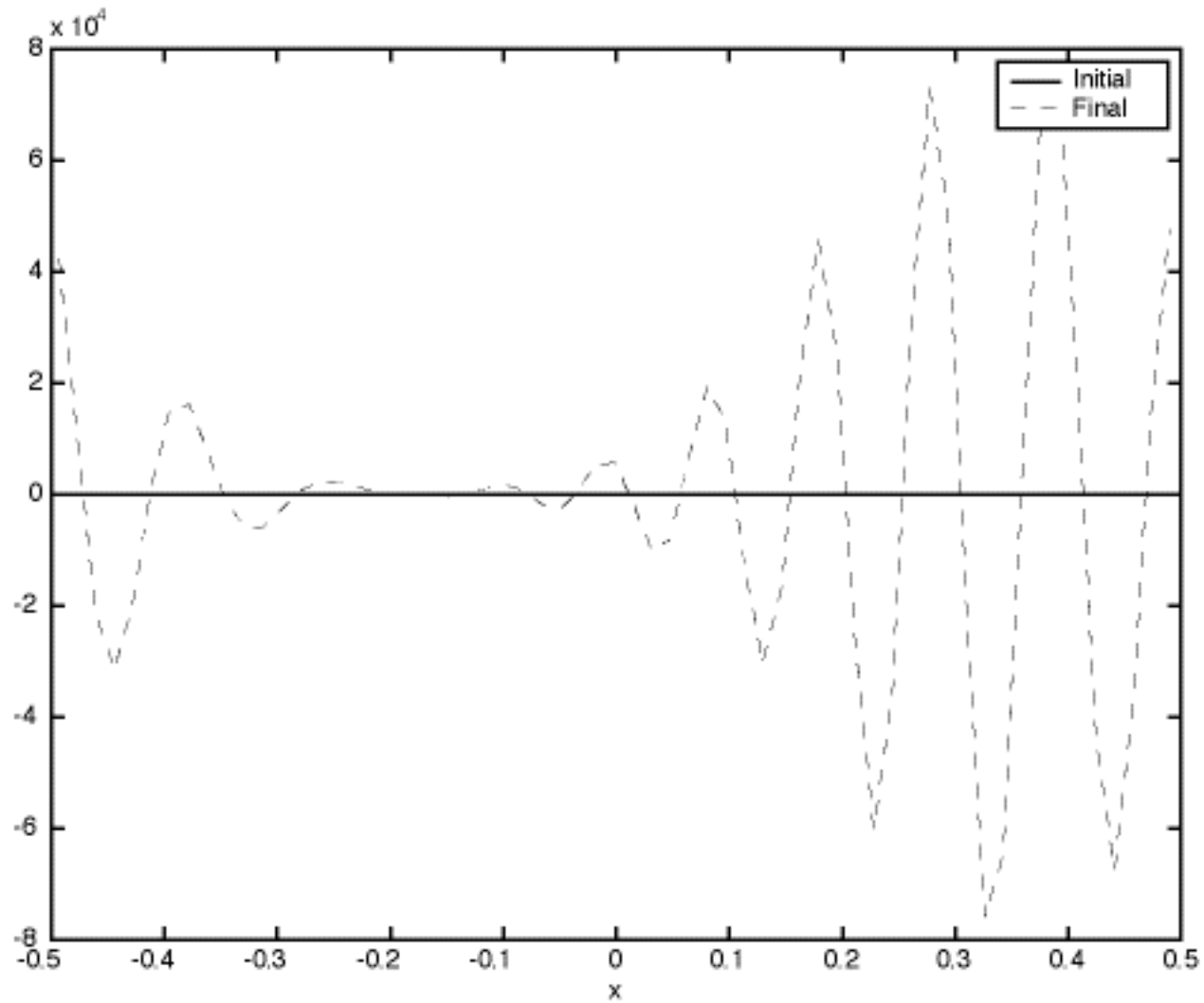
$$A \equiv \frac{T^{n+1}}{T^n} = 1 - \frac{c\tau}{2h} (e^{ikh} - e^{-ikh})$$

that becomes

$$A^2 = \left[ 1 + \left( \frac{c\tau}{h} \right)^2 \sin^2 kh \right]$$

Which is unconditionally unstable.

FTCS  $N=61$ , 62.5 steps,  $\tau=0.016$ , CFL=1



# Lax Method for the Advection Equation

If we replace the term  $a_i^n$  with an average of the values around it, i.e.:

$$a_i^n \rightarrow \frac{a_{i+1}^n + a_{i-1}^n}{2}$$

we then get

$$a_i^{n+1} = \frac{1}{2}(a_{i+1}^n + a_{i-1}^n) - \frac{c\tau}{2h}(a_{i+1}^n - a_{i-1}^n)$$

This is the *LAX* method and is stable for values

$$\tau < \frac{h}{c}$$

which is known as the *Courant-Friedrichs-Lewy (CFL)* condition.

- It says that the velocity that information propagates in the system should be faster than the velocity of the solution.

# Lax Method for the Advection Equation

The averaging term in the Lax method succeeds in stabilizing the scheme by adding diffusion

- the magnitude of the diffusion is inversely proportional to the timestep
- So that if  $\tau$  is too small the diffusion dominates the solution
- If  $\tau$  is too large the diffusion is too weak to stabilize the solution

# Von Neumann Stability analysis for the Lax method

Again assume  $a_j^n = T^n e^{ikjh}$

So that the Lax method becomes

$$T^{n+1} e^{ikjh} = \frac{T^n e^{ik(j+1)h} + T^n e^{ik(j-1)h}}{2} - \frac{c\tau}{2h} T^n (e^{ik(j+1)h} - e^{ik(j-1)h})$$

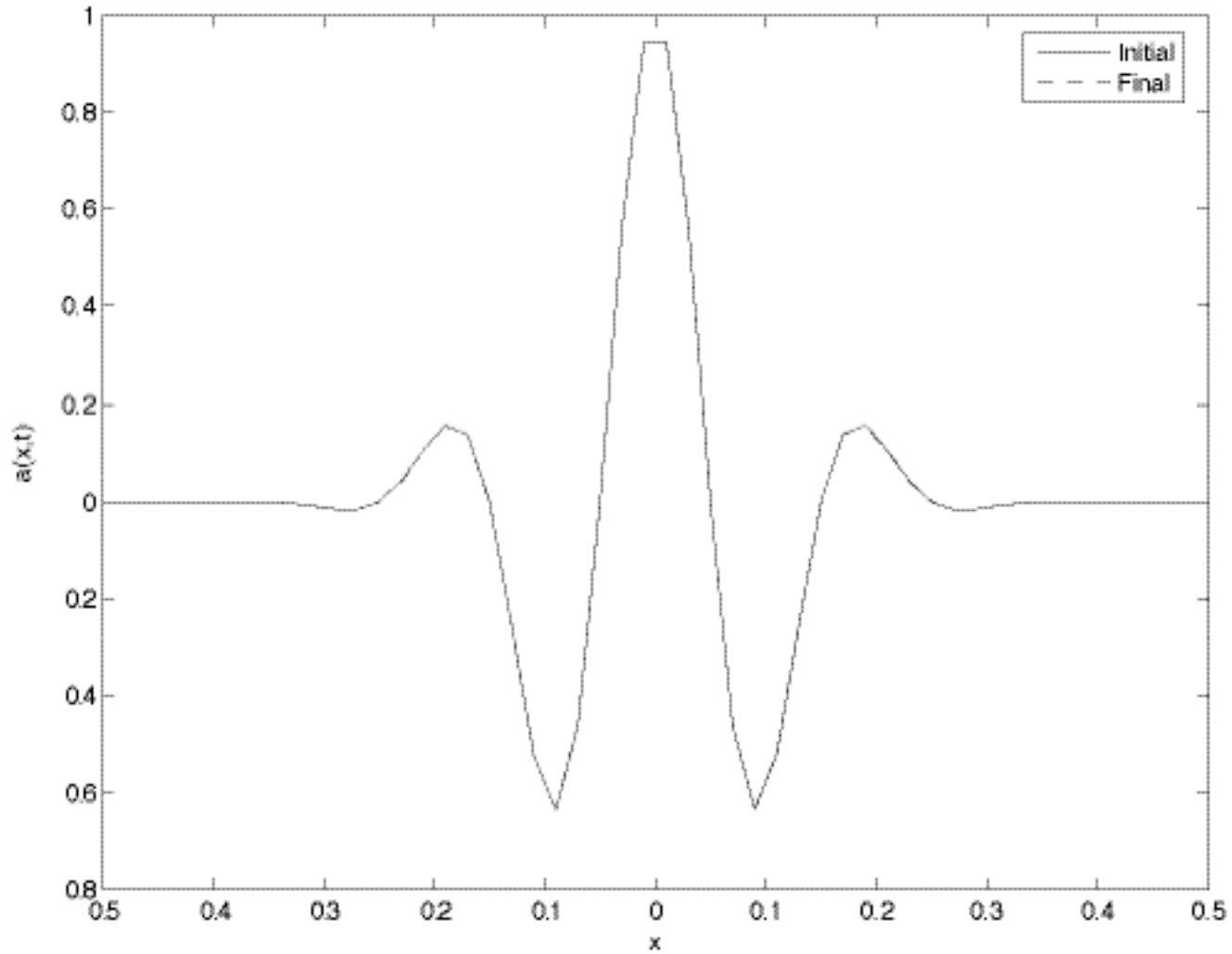
Or  $A \equiv \frac{T^{n+1}}{T^n} = \frac{(e^{ikh} + e^{-ikh})}{2} - \frac{c\tau}{2h} (e^{ikh} - e^{-ikh})$

that becomes  $A^2 = \left[ \cos^2 kh + \left( \frac{c\tau}{h} \right)^2 \sin^2 kh \right]$

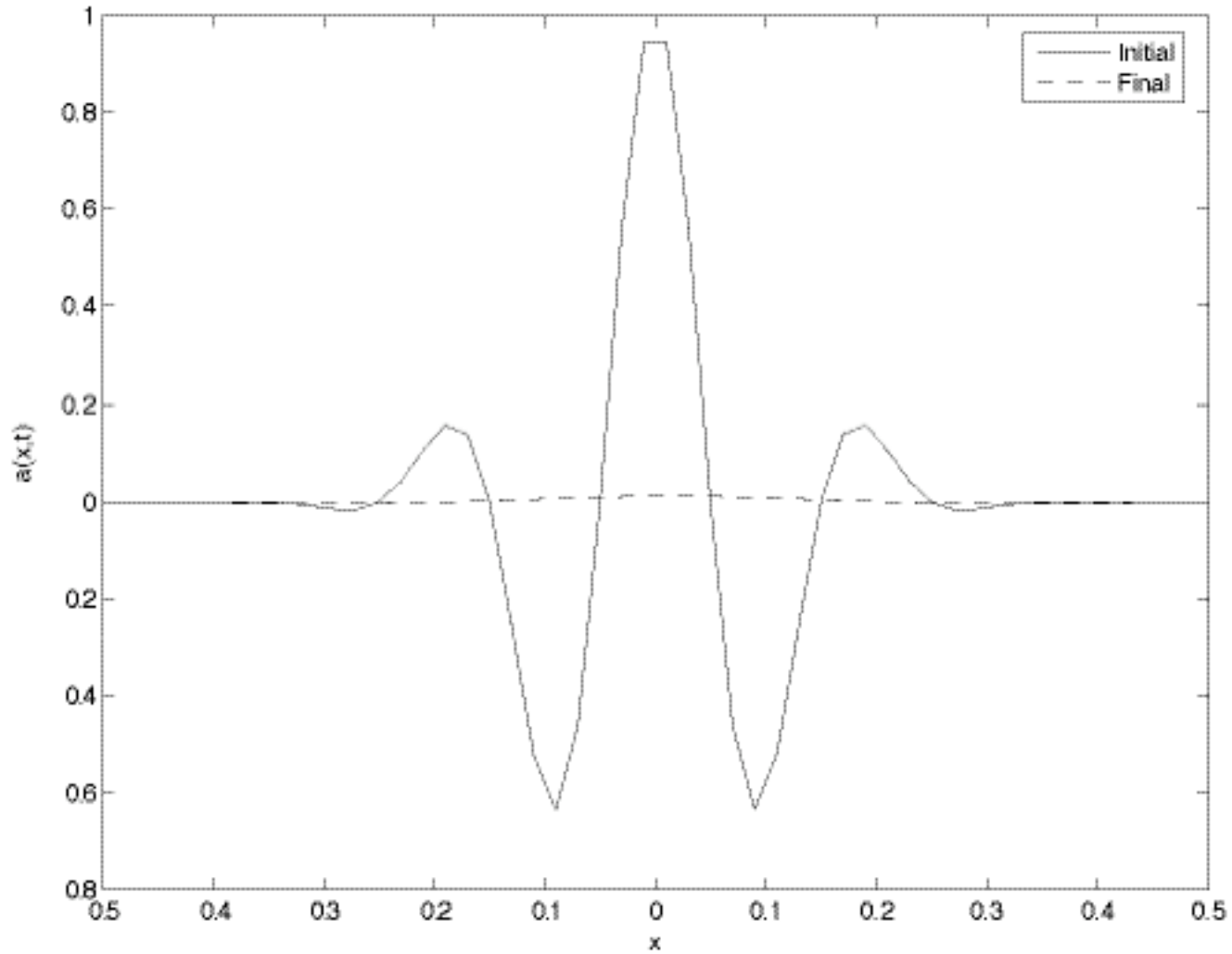
Which is stable when  $\tau \leq \frac{h}{c}$

Note that when,  $\tau = \frac{h}{c}$   $|A| = 1$

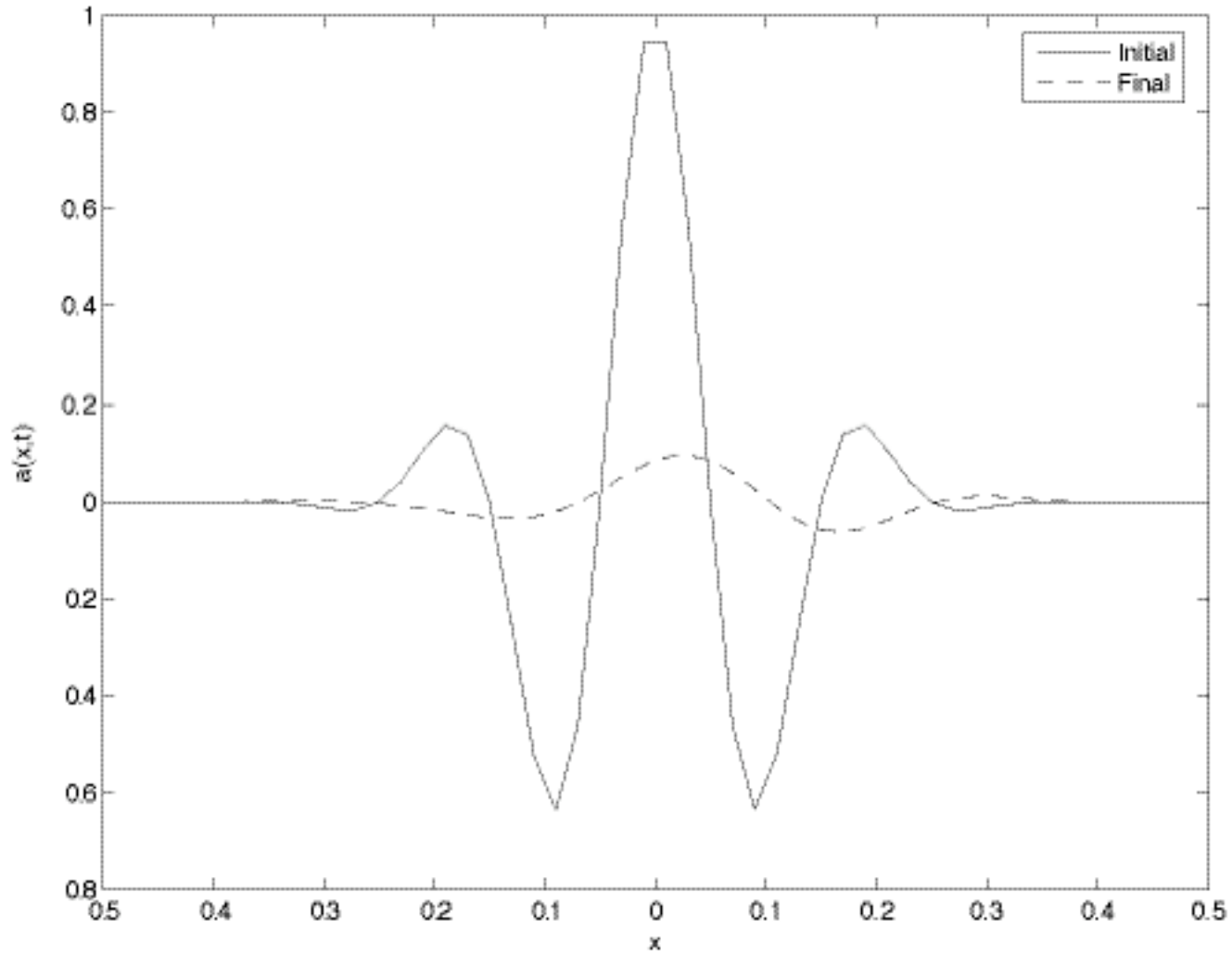
Lax  $N=50$ , 50 steps,  $\tau=0.02$ , CFL=1



Lax  $N=50$ , 100 steps,  $\tau=0.01$ , CFL=0.5



Lax  $N=50$ , 60 steps,  $\tau=0.016$ , CFL=0.83



# Phase Error

Another error that arises from the numerical approximation is a phase error. For the simple wave equation, the exact solution is a wave that moves to the right at speed  $c$ . Numerical solutions will typically change the amplitude and the phasing of the wave, the amount will vary depending on the wavenumber ( $k$ ) of the wave. The amplification factor  $A$  can be written as

$$A = |A|e^{i\theta}$$

Where  $|A| = \sqrt{\text{Re}(A)^2 + \text{Im}(A)^2}$

$$\theta = -\tan^{-1}\left(\frac{\text{Im}(A)}{\text{Re}(A)}\right)$$

# Phase Error

If we had an initial condition

$$\Psi(x,0) = \sum a_k e^{ikx}$$

Where  $k$  is the wavenumber and  $a_k$  is the  $k$  fourier amplitude, then at some later time  $t$ , the exact solution takes the form

$$\Psi(x,t) = \sum_k a_k e^{i(kx-ct)}$$

Which is simply

$$\begin{aligned}\Psi(x,t) &= e^{-ict} \sum_k a_k e^{ikx} \\ &= e^{-ict} \Psi(x,0)\end{aligned}$$

That simply states that the wave has moved a distance  $ct$ . To be exact, a numerical solution should have the same phase shift. So a measure of the phase shift error would then be how close to 1 is ratio

$$\frac{\theta}{kct}$$

# Upwind Method for the Advection Equation

Since information is coming from the upstream direction, a physically meaningful approach is to use upwind

information only 
$$\frac{a_i^{n+1} - a_i^n}{\tau} = -c \frac{a_i^n - a_{i-1}^n}{h}$$

we then get

$$a_i^{n+1} = a_i^n - \frac{c\tau}{h} (a_i^n - a_{i-1}^n)$$

# Leap Frog Method for the Advection Equation

To get second order in time we can use the following

we then get 
$$\frac{a_i^{n+1} - a_i^{n-1}}{2\tau} = -c \frac{a_{i+1}^n - a_{i-1}^n}{2h}$$

$$a_i^{n+1} = a_i^{n-1} - \frac{c\tau}{h} (a_{i+1}^n - a_{i-1}^n)$$

The difficulty here is that 2 previous time levels have to be kept, requiring more memory and the code is not self-starting. Again, this method and is also stable for values, but has the advantage that  $|A|=1$ .

$$\tau < \frac{h}{c}$$

# Lax Wendroff Method for the Advection Equation

Another popular method is the Lax Wendroff scheme. It can be derived from the Taylor series expansion for  $a(x, t + \tau)$

$$a(x, t + \tau) = a(x, t) + \tau \frac{\partial a}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 a}{\partial t^2} + O(\tau^3)$$

using a general form of the Advection equation

$$\frac{\partial a(x, t)}{\partial t} = - \frac{\partial F(a)}{\partial x}$$

Where  $F(a)$  is a *FLUX* function. Differentiating the above wrt  $t$

But 
$$\frac{\partial^2 a(x, t)}{\partial t^2} = - \frac{\partial}{\partial t} \frac{\partial F(a)}{\partial x} = - \frac{\partial}{\partial x} \frac{\partial F(a)}{\partial t}$$

$$\frac{\partial F(a)}{\partial t} = \frac{dF(a)}{da} \frac{\partial a}{\partial t} = F'(a) \frac{\partial a}{\partial t} = -F'(a) \frac{\partial F}{\partial x}$$

# Lax Wendroff Method

Going back to

$$a(x, t + \tau) = a(x, t) + \tau \frac{\partial a}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 a}{\partial t^2} + O(\tau^3)$$

and using

$$\frac{\partial^2 a}{\partial t^2} = - \frac{\partial}{\partial x} \frac{\partial F(a)}{\partial t}$$

and

$$\frac{\partial a}{\partial t} = - \frac{\partial F(a)}{\partial x}$$

$$\frac{\partial F(a)}{\partial t} = -F'(a) \frac{\partial F}{\partial x}$$

we get

$$a(x, t + \tau) = a(x, t) - \tau \frac{\partial F}{\partial x} + \frac{\tau^2}{2} \frac{\partial}{\partial x} \left( F'(a) \frac{\partial F}{\partial x} \right) + O(\tau^3)$$

# Lax Wendroff Method - Finite difference form

In finite difference form, we have

$$\begin{aligned} a_i^{n+1} &= a_i^n - \tau \frac{F_{i+1} - F_{i-1}}{2h} + \frac{\tau^2}{2h} \left( \left[ F'(a) \frac{\partial F}{\partial x} \right]_{i+\frac{1}{2}} - \left[ F'(a) \frac{\partial F}{\partial x} \right]_{i-\frac{1}{2}} \right) \\ &= a_i^n - \tau \frac{F_{i+1} - F_{i-1}}{2h} + \frac{\tau^2}{2h^2} \left( F'_{i+\frac{1}{2}}(a)(F_{i+1} - F_i) - F'_{i-\frac{1}{2}}(a)(F_i - F_{i-1}) \right) \end{aligned}$$

where

$$F'_{i\pm\frac{1}{2}} = F' \left( \left[ \frac{a_{i\pm 1}^n + a_i^n}{2} \right] \right)$$

# Lax Wendroff Method

For the PDE

$$\frac{\partial a(x,t)}{\partial t} = -c \frac{\partial a(x,t)}{\partial x}$$

so that we have  $F_i = ca_i^n$        $F'_i = c$

$$a_i^{n+1} = a_i^n - \frac{c\tau}{2h} (a_{i+1}^n - a_{i-1}^n) + \frac{c^2\tau^2}{2h^2} (a_{i+1}^n - 2a_i^n + a_{i-1}^n)$$

the CFL condition for this is also  $\tau < \frac{h}{c}$

And when  $\tau = \frac{h}{c}$  then for the simple advection problem, which is the exact solution

$$a_i^{n+1} = a_{i-1}^n$$

# Von Neumann Stability analysis for the Lax Wendroff method

Again assume

$$a_j^n = T^n e^{ikjh}$$

So that the Lax Wendroff method becomes

$$T^{n+1} e^{ikjh} = T^n e^{ikjh} - \frac{c\tau}{2h} T^n (e^{ik(j+1)h} - e^{ik(j-1)h}) + \frac{1}{2} \left( \frac{c\tau}{h} \right)^2 T^n (e^{ik(j+1)h} - 2 + e^{ik(j-1)h})$$

Or 
$$A = \frac{T^{n+1}}{T^n} = 1 - \frac{c\tau}{2h} (e^{ikh} - e^{-ikh}) + \frac{1}{2} \left( \frac{c\tau}{h} \right)^2 (e^{ikh} - 2 + e^{-ikh})$$

that becomes 
$$A = \frac{T^{n+1}}{T^n} = 1 - i \frac{c\tau}{2h} \sin kh - 2 \left( \frac{c\tau}{h} \right)^2 \sin^2 \left( \frac{kh}{2} \right)$$

$$|A|^2 = 1 - 4 \left( \frac{c\tau}{h} \right)^2 \left( 1 - \left( \frac{c\tau}{h} \right)^2 \right) \sin^4 \left( \frac{kh}{2} \right)$$

Which is stable when 
$$\tau \leq \frac{h}{c}$$

Lax Wendroff N=50, 60 steps,  $\tau=0.016$ , CFL=0.83

