High Resolution Methods for Conservation Equations

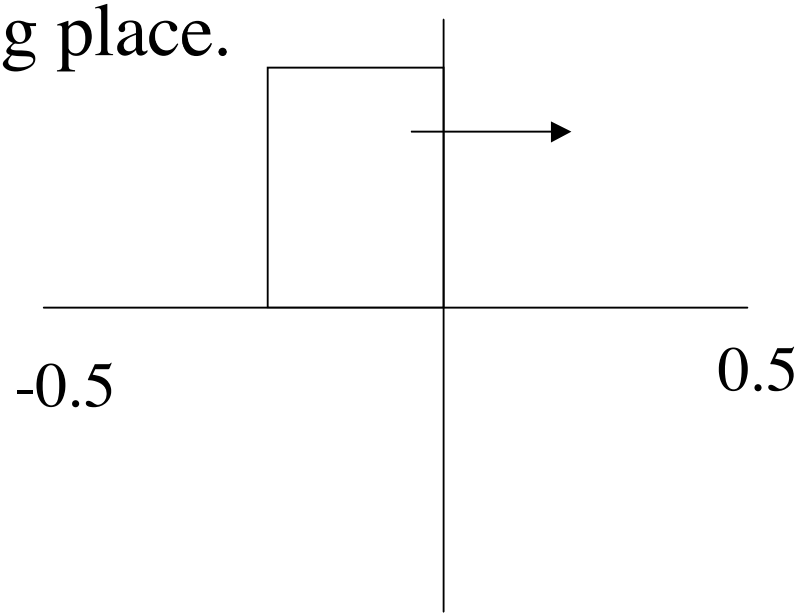
You may have noticed that in some cases the traffic program produced spurious oscillations that in come cases were negative. To understand this, lets go back to the simple wave equation that we looked at earlier, ie.,

$$\frac{\partial u}{\partial t} = -c\,\frac{\partial u}{\partial x}$$

[Ref: Finite Volume methods for Hyperbolic problems, R. LeVeque, Cambridge Press, 2002]

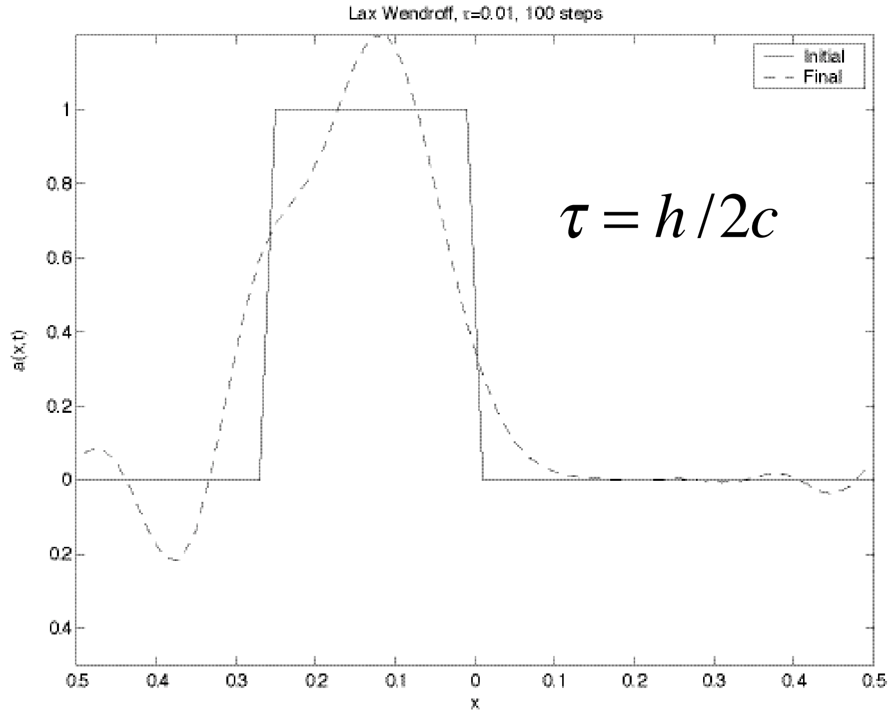
Square Wave Example

Lets look at what the numerical method does in the case of a simple square wave moving to the right at speed 1 in a periodic system of length . We will run the code until the waveform returns to its starting place.

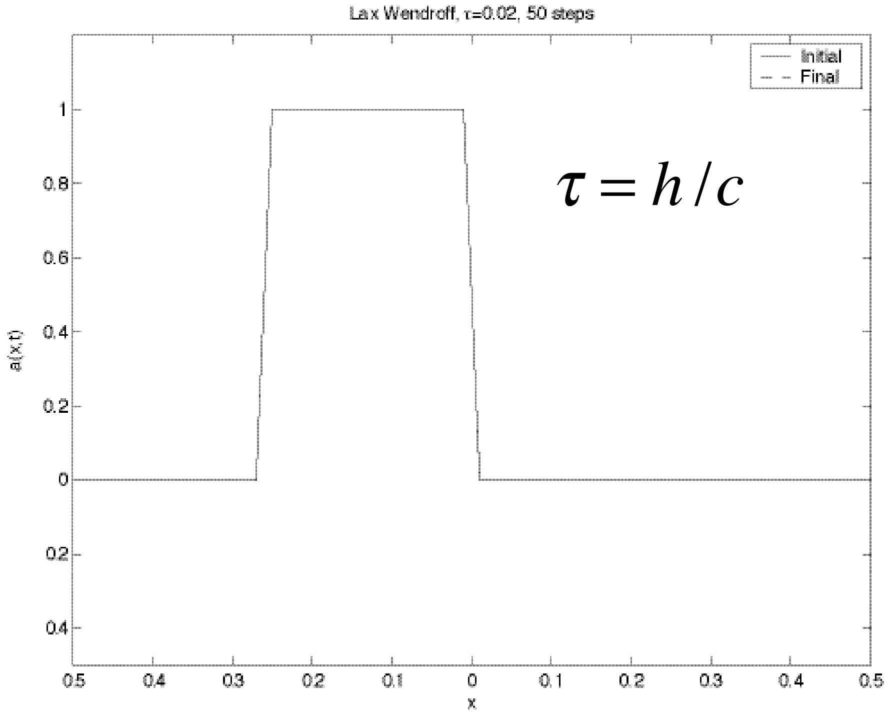


Lax Wendroff Solution

This shows the computed numerical solution (dashed) plotted against the exact solution for 1 period using the `advect` program



$\tau = h/2c$



$\tau = h/c$

Lax Wendroff

When the timestep is half the courant number, the solution is inaccurate

But when $\tau = h/c$ Let Q_i^n represent the numerical solution.

We can write out the scheme explicitly as

$$Q_i^{n+1} = Q_i^n - \frac{c\tau}{2h}(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}\left(\frac{c\tau}{h}\right)^2 (Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n)$$

When $\tau = h/c$ the above simply reduces to $Q_i^{n+1} = Q_{i-1}^n$

Which simply says that each cell is replaced with the value on the cell on its left.

Upwind Scheme

An even simpler scheme is the upwind scheme

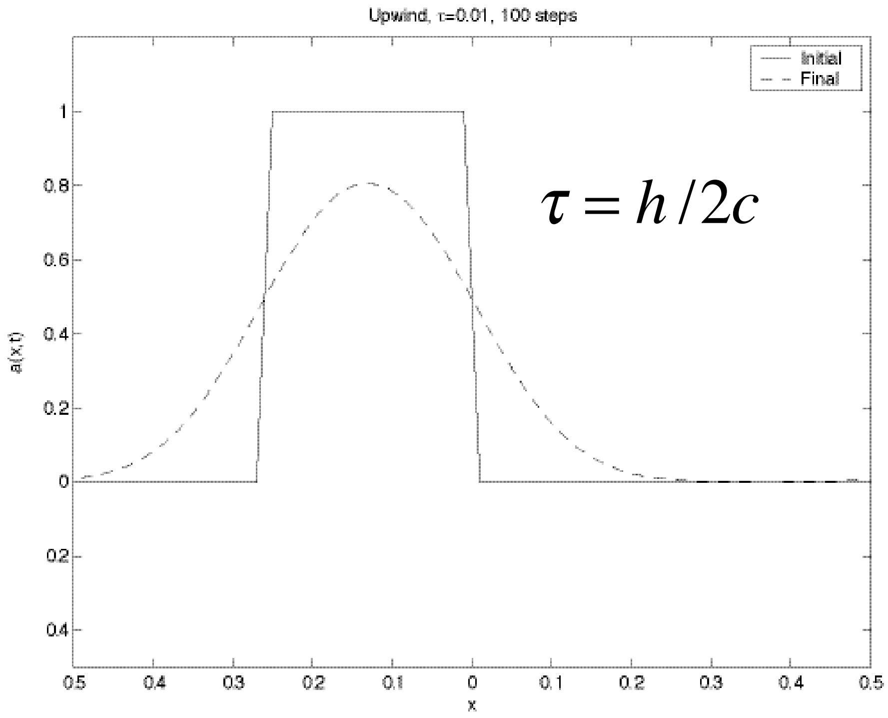
$$Q_i^{n+1} = Q_i^n - \frac{c\tau}{h}(Q_i^n - Q_{i-1}^n)$$

Again for the case where $\tau = h/c$ we get $Q_i^{n+1} = Q_{i-1}^n$

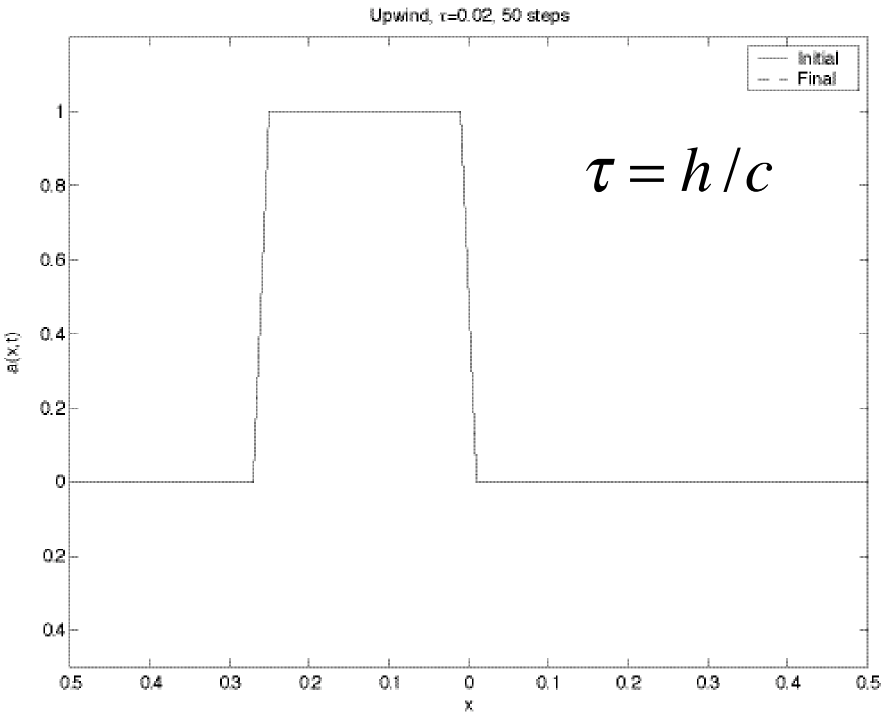
But when $\tau = h/2c$ we get that $Q_i^{n+1} = \frac{1}{2}(Q_i^n + Q_{i-1}^n)$

The results are illustrated on the next page.

Upwind Scheme



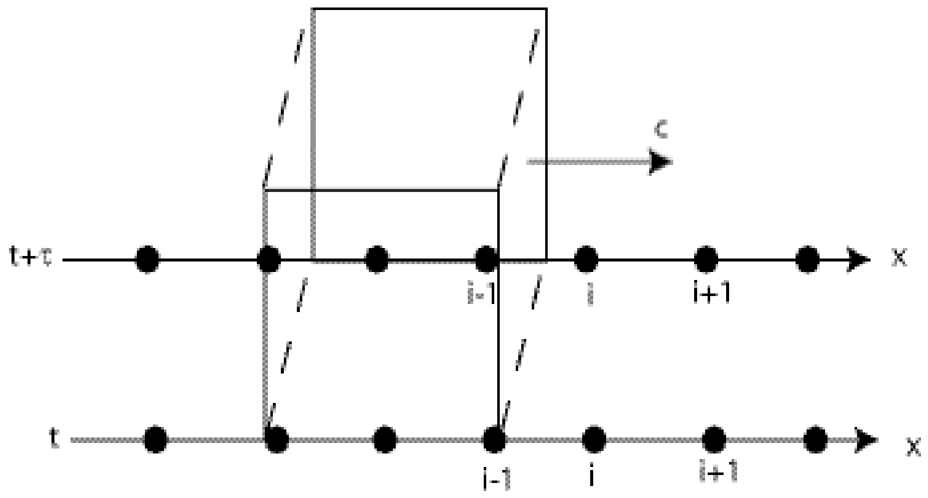
$\tau = h/2c$



$\tau = h/c$

Alternative Interpretation

Another way of looking at these schemes is in terms of characteristics



REA

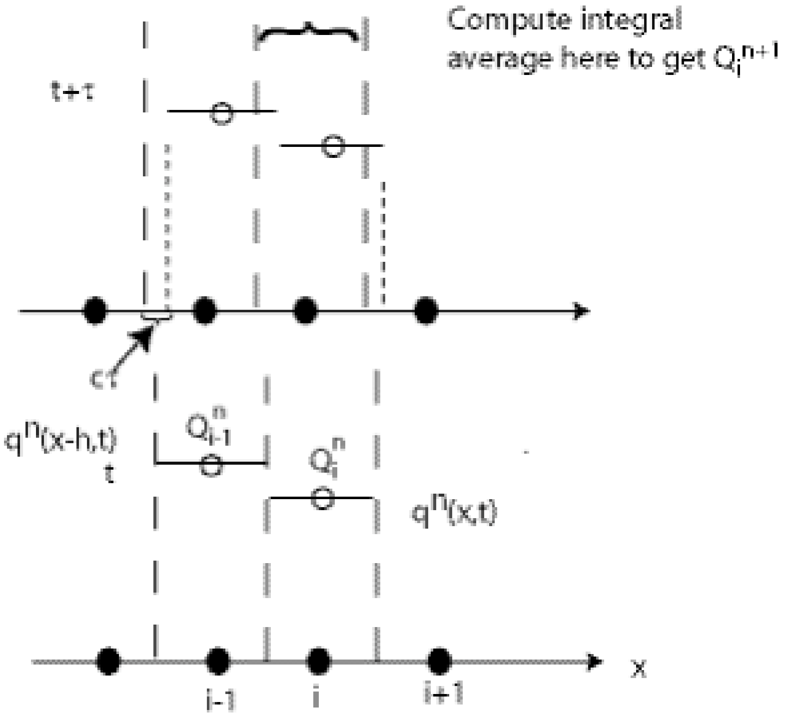
The upwind method for the advection equation is a special case of the so-called REA algorithm, which stands for Reconstruct, Evolve and Average.

Reconstruct: Each numerical quantity Q_i^n on the grid can be interpreted as the average of a function $q^n(x,t)$ over the grid cell.

Evolve: Move the function $q^n(x,t+\tau)$ a distance ut

Average: Compute the average of $q^n(x,t+\tau)$ to compute the new Q_i^{n+1}

REA - Upwind method



The Upwind method-Piecewise constant

Reconstruct: We interpret the values stored at each grid point as some characteristic average value of the quantity over the cell and if we assume that the values are stored as step functions $q_i^n(x)$ (piecewise constant)

Evolve: Move the functions $q_i^n(x)$ a distance $c\tau$

Average: The the new value Q_i^{n+1} by computing the average value over the cell from i-1/2 to i+1/2

$$Q_i^{n+1} = \frac{1}{h}((h - c\tau)Q_i^n + u\tau Q_{i-1}^n)$$

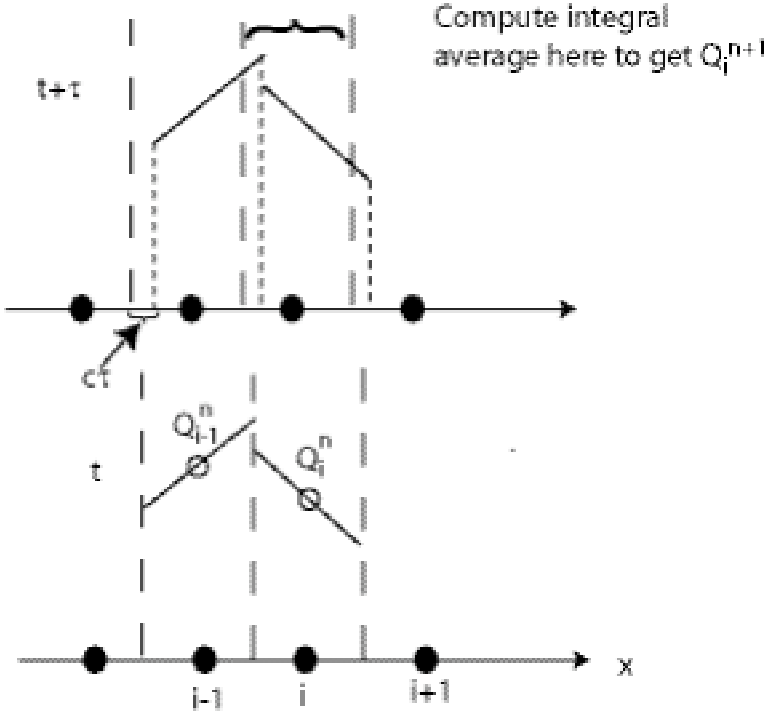
$$= Q_i^n - \frac{c\tau}{h}(Q_i^n - Q_{i-1}^n)$$

Piecewise Linear

If we generalize the approximation and assume that the values vary linearly between grid points with a function of slope σ_i, then in the range $[i-\frac{1}{2}, i+\frac{1}{2}]$

The function takes on the form

$$q_i^n(x) = Q_i^n + \sigma_i^n(x - x_i)$$



Piecewise Linear (cont.)

$$Q_i^{n+1} h = \int_{x_i - h/2 - u\tau}^{x_i - h/2} q_{i-1}^n(x)\,dx + \int_{x_i - h/2}^{x_i + h/2 - u\tau} q_i^n(x)\,dx$$

$$Q_i^{n+1} h = \int_{x_i - h/2 - u\tau}^{x_i - h/2} Q_{i-1}^n + \sigma_{i-1}^n (x - x_{i-1})\,dx + \int_{x_i - h/2}^{x_i + h/2 - u\tau} Q_i^n + \sigma_i^n (x - x_i)\,dx$$

$$Q_i^{n+1} = \frac{c\tau}{2h}[2Q_{i-1}^n + \sigma_{i-1}^n(x_i - \frac{h}{2} - x_{i-1} + x_i - \frac{h}{2} - c\tau - x_{i-1})] + \frac{h - c\tau}{2h}[2Q_i^n + \sigma_i^n(\frac{h}{2} - c\tau - \frac{h}{2})]$$

$$= \frac{c\tau}{h}[Q_{i-1}^n + \frac{1}{2}\sigma_{i-1}^n(h - c\tau)] + \frac{h - c\tau}{h}[Q_i^n - \frac{1}{2}\sigma_i^n c\tau]$$

$$= Q_i^n - \frac{c\tau}{h}(Q_i^n - Q_{i-1}^n) - \frac{c\tau}{2h}(h - c\tau)(\sigma_i^n - \sigma_{i-1}^n) \qquad (1)$$

This is basically an upwind method with a correction that depends on the slopes.

However, one has a choice of slopes

Upwind: $\sigma_i^n = 0$

Lax-Wendroff: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_i^n}{h}$ (Downwind)

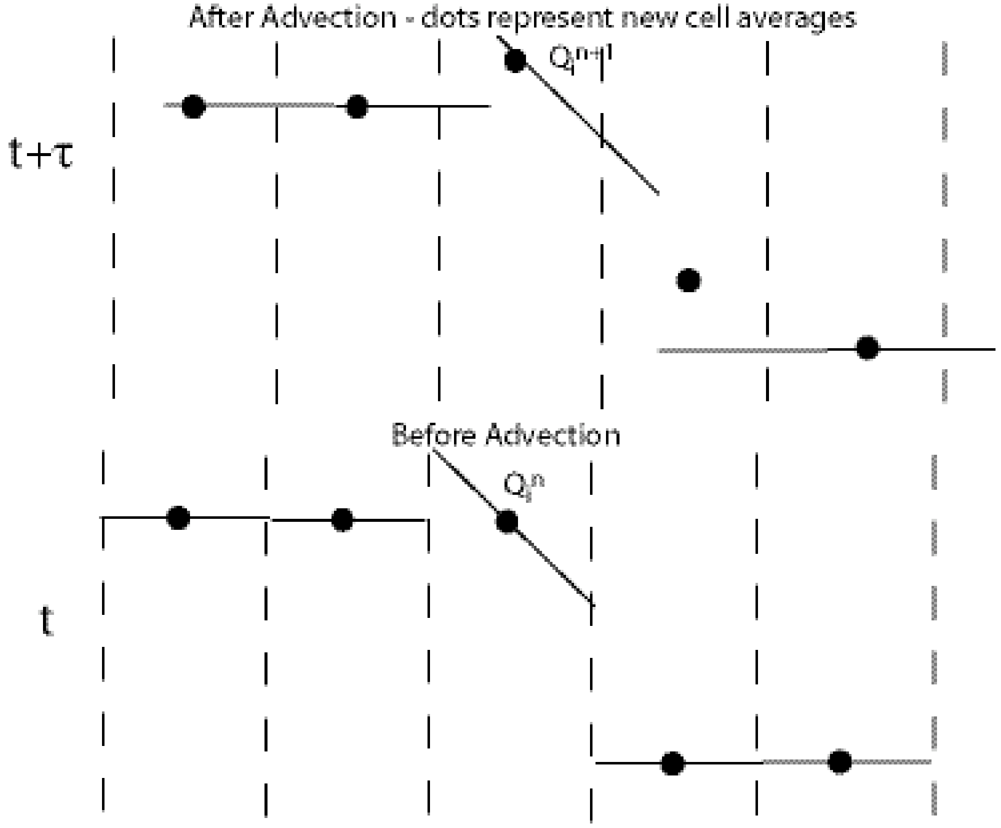
Beam-Warming: $\sigma_i^n = \dfrac{Q_i^n - Q_{i-1}^n}{h}$ (Upwind)

Fromm: $\sigma_i^n = \dfrac{Q_{i+1}^n - Q_{i-1}^n}{2h}$ (Centered)

The last 3 choices of slopes give the formal order of accuracy, but only the downwind version results in a symmetric formula.

Problem with Lax Wendroff

Lets look at the Lax Wendroff method when it encounters a discontinuity - note the overshoot →



Limiters

Another way to write equation (1) is in the form

$$Q_i^{n+1} = Q_i^n - \frac{\tau}{h}(F_{i+1/2}^n - F_{i-1/2}^n)$$

where $F_{i\pm1/2}^n$ corresponds to the flux values that move across cell interfaces.

The idea behind the limiters is to limit the amount of flux to prevent undershoots and overshoots.

Slope Limiters

The trick is to limit the slope σ_i^n to reduce artifacts such as overshoots while at the same time minimizing diffusion. To proceed, we need to generalize our expressions for the flux to take into account all signs of the velocity u (so far we have implicitly assumed that u >0). We can then write the flux as

$$F_{i-1/2}^n = \begin{cases} cQ_{i-1}^n + \tfrac{1}{2}c(h - c\tau)\sigma_{i-1}^n & \text{if } u \geq 0 \\ cQ_i^n - \tfrac{1}{2}c(h + c\tau)\sigma_i^n & \text{if } u \leq 0 \end{cases}$$

minmod Slope Limiters

Slope limiters take various function forms, each having advantages and disadvantages. One popular one is the so called minmod limiter, where

$$\sigma_i^n = \min\mathrm{mod}(\frac{Q_i^n - Q_{i-1}^n}{h}, \frac{Q_{i+1}^n - Q_i^n}{h})$$

and

$$\min\mathrm{mod}(a,b) = \begin{cases} a & \text{if } |a| < |b| \quad \text{and } \ ab > 0 \\ b & \text{if } |a| > |b| \quad \text{and } \ ab > 0 \\ 0 & \text{if } \ ab < 0 \end{cases}$$

If a and b have the same sign, it chooses the one with the smallest absolute value. If the signs are opposite, then we have a local maxima or minima, in which case the slope is set to zero and diffusion is turned on to ‘clean’ it out.

Total Variation Diminishing (TVD)

So what does it mean to ‘clean’ up a solution? Essentially we need a measure of the amount of oscillations in the system, a measure of which is known as the total variation of a function

$$TV(Q) = \sum_i \left| Q_i - Q_{i-1} \right|$$

What is want is that TV(Q) be constant in time. The following schemes can be shown to have this property.

Flux Limiters

Rather than using the slope we will look at the jumps across cell faces where we will have the jump

$$\Delta Q_{i-1/2}^{n} = Q_i^n - Q_{i-1}^n$$

So a more general way to write the flux is

$$F_{i-1/2}^{n} = c^{+}Q_{i-1}^{n} + c^{-}Q_{i}^{n} + \tfrac{1}{2}\left|c\right|(1 - \left|\frac{c\tau}{h}\right|)\delta_{i-1/2}^{n}$$

where u is the average velocity in cell i and

$c^{+} = \max(c,0)$ $c^{-} = \min(c,0)$

$\delta_{i-1/2}^{n} =$ a limited version of $\Delta Q_{i-1/2}^{n}$

Flux Limiters (cont.)

We will write $\delta_{i-1/2}^{n} = \phi(\theta_{i-1/2}^{n})\, \Delta Q_{i-1/2}^{n}$

where $$\theta_{i-1/2}^{n} = \frac{\Delta Q_{I-1/2}^{n}}{\Delta Q_{i-1/2}^{n}}$$

and $$I = \begin{cases} i-1 & \text{if} \quad u > 0 \\ i+1 & \text{if} \quad u < 0 \end{cases}$$

Linear Methods

upwind: $\phi(\theta) = 0$

Lax-Wendroff: $\phi(\theta) = 1$

Beam-Warming: $\phi(\theta) = \theta$

Fromm: $\phi(\theta) = \frac{1}{2}(1 + \theta)$

High Resolution limiters

minmod: $\phi(\theta) = \min\mathrm{mod}(1, \theta)$

where

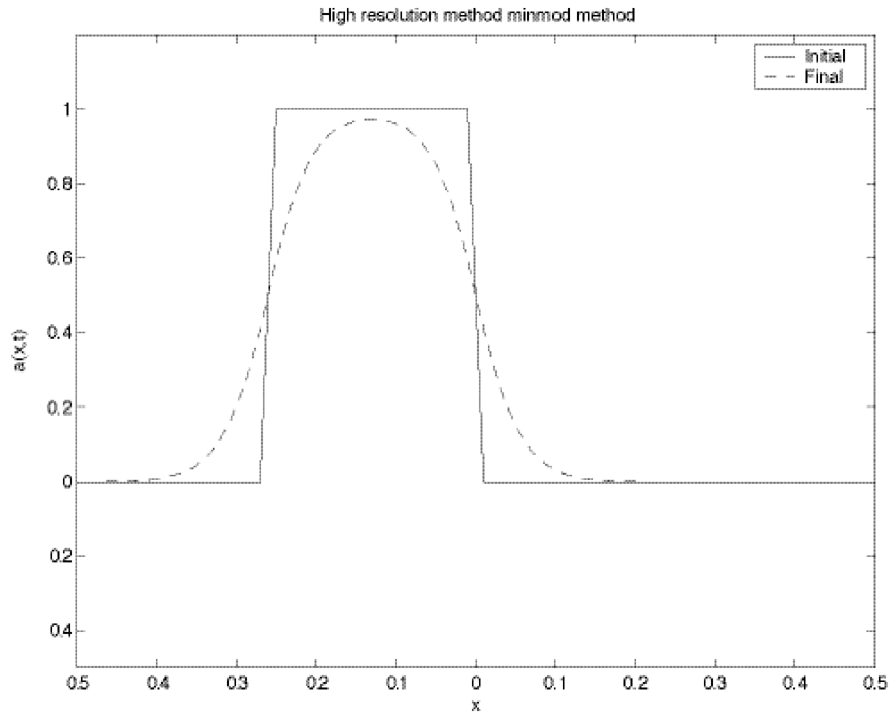
$$\min\mathrm{mod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \quad \text{and } ab > 0 \\ b & \text{if } |a| > |b| \quad \text{and } ab > 0 \\ 0 & \text{if } ab < 0 \end{cases}$$

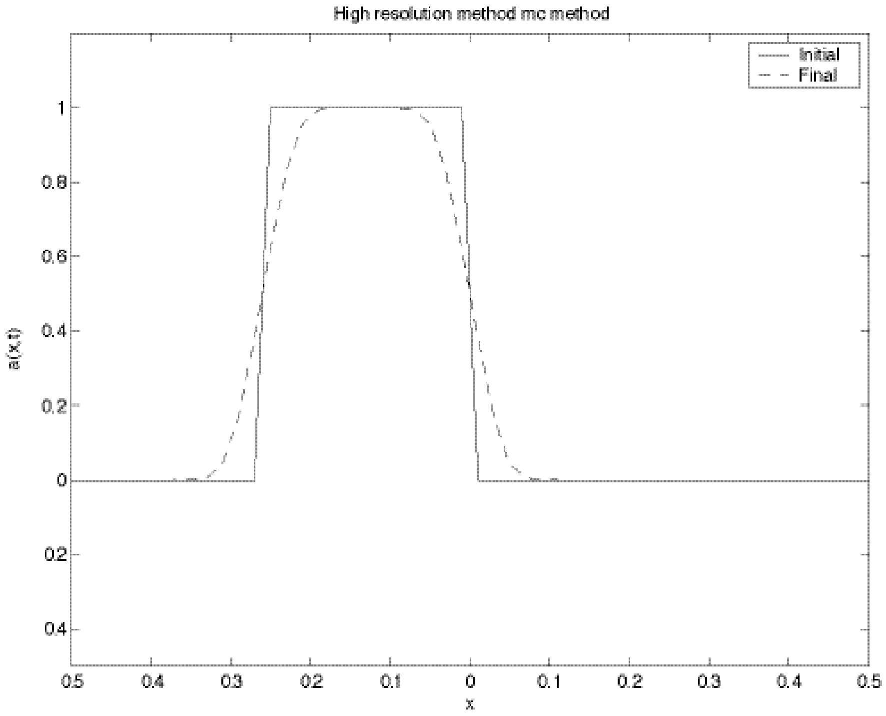
superbee: $\phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta))$

MC: $\phi(\theta) = \max(0, \min(1 + \theta)/2, 2, 2\theta))$

van Leer: $\phi(\theta) = \dfrac{\theta + |\theta|}{1 + |\theta|}$

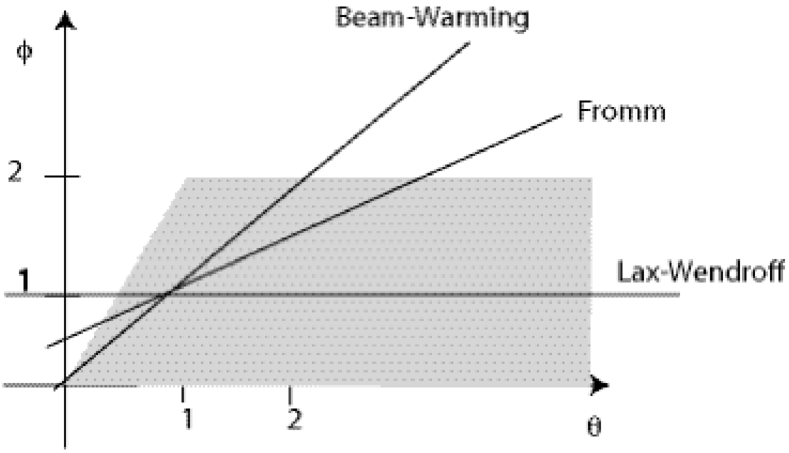
Two high resolution methods for τ=0.01 (=h/2c)



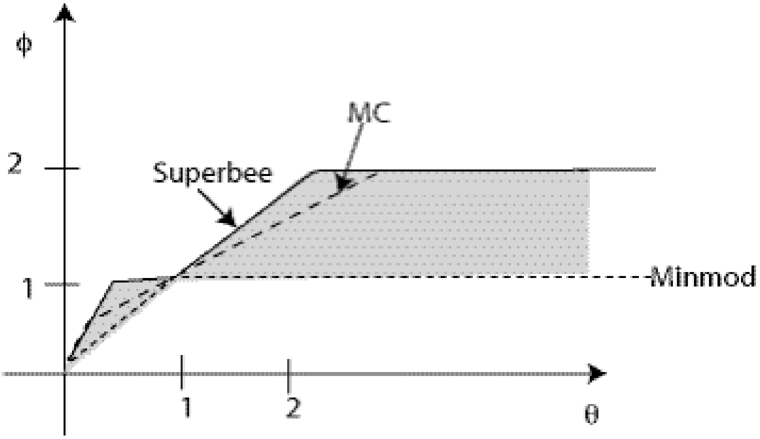


Methods give exact answer for τ=0.02 (=h/c)

One can show that for the scheme to be TVD, it must line inside the shaded region



The shaded region shows where the schemes must lie to be TVD



The shaded region shows where the schemes must lie to be TVD