

ODEs: Basic Methods

PHYS 416

Garcia, Chapter 2

Projectile Motion

The Basic Equations for 2D projectile motion are

$$\frac{d\vec{r}}{dt} = \vec{v}$$

Where

$$\frac{d\vec{v}}{dt} = -g\hat{y} + \frac{1}{m}\vec{F}_a(\vec{v}) \quad (1)$$

m = mass (kg)

g = acceleration due to gravity ($\approx 9.81 \text{ m/s}^2$)

\vec{r} = position (x,y)

\vec{v} = velocity (x,y)

$\vec{F}_a(\vec{v})$ = air resistance force

Projectile motion - no air resistance

In the simple case of an object launched with initial speed v_0 and an angle θ with respect to the ground at a height h_0 , one can show that the time of flight t_R is

$$t_R = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gy_0}}{g}$$

the range is $x_{\max} = v_0 t_R \cos \theta$

and the maximum height is $y_{\max} = y_0 + \frac{v_0^2 \sin^2 \theta}{2g}$

Air Resistance

The air resistance term takes on the form

$$\vec{F}_a(\vec{v}) = -\frac{1}{2}C_d\rho A|\vec{v}|\vec{v}$$

where

ρ = density of air (in kg/m^3)

A = cross sectional area of projectile (in m^2)

C_d = drag coefficient (dimensionless)

which depends speed and the geometry of
the object

Drag Coefficient

The drag coefficient for a sphere (radius R) moving slowly through the air is defined by Stoke's Law as

$$C_d = \frac{12\nu}{Rv} = \frac{24}{R_e}$$

Where ν = viscosity of the air ($\approx 1.5 \times 10^{-5} \text{ m}^2/\text{s}$ for air)

R_e = Reynolds Number (dimensionless)

$$= \frac{2Rv}{\nu}$$

ν = viscosity of the air ($\approx 1.5 \times 10^{-5} \text{ m}^2/\text{s}$)

Drag Coefficient - Reynolds Number

For baseball, Stokes Law is valid for $v < 0.2$ mm/s

$$(R_e \approx 1)$$

When $R_e > 10^3$, there is a wake that forms behind the object which results in a roughly constant $C_d \approx 0.5$

When $R_e > 10^5$, turbulence destroys the wake and C_d drops dramatically

Numerical Methods

We will begin by using the simplest numerical method, known as the Euler Method.

If we divide time into a set of discrete intervals (τ) so that at the n th interval the time is $t = (n-1)\tau$ We then convert equations (1) into finite difference approximations

$$\frac{\vec{r}(t + \tau) - \vec{r}(t)}{\tau} \approx \vec{v}(t)$$

$$\frac{\vec{v}(t + \tau) - \vec{v}(t)}{\tau} \approx \vec{a}(t)$$

Rearranging, we get

$$\vec{r}(t + \tau) \approx \vec{r}(t) + \tau\vec{v}(t)$$

$$\vec{v}(t + \tau) \approx \vec{v}(t) + \tau\vec{a}(t)$$

Which provides a simple prescription of getting the new position at time $t+\tau$ given the information at time t . This approximation introduces a certain amount of error, and we can rewrite the above equation as

$$\vec{r}(t + \tau) = \vec{r}(t) + \tau\vec{v}(t) + O(\tau^2) \tag{2}$$

$$\vec{v}(t + \tau) = \vec{v}(t) + \tau\vec{a}(t) + O(\tau^2)$$

Where the $O(\tau^2)$ is an error term that depends on the timestep τ .

Other Simple Methods

Introducing a simpler notation, $\vec{r}_n = \vec{r}(t) = \vec{r}((n-1)\tau)$

$$\left. \begin{aligned} \vec{r}_{n+1} &\approx \vec{r}_n + \tau \vec{v}_n \\ \vec{v}_{n+1} &\approx \vec{v}_n + \tau \vec{a}_n \end{aligned} \right\}$$

Euler Method

$$\left. \begin{aligned} \vec{v}_{n+1} &\approx \vec{v}_n + \tau \vec{a}_n \\ \vec{r}_{n+1} &\approx \vec{r}_n + \tau \vec{v}_{n+1} \end{aligned} \right\}$$

Euler Cromer Method

$$\left. \begin{aligned} \vec{v}_{n+1} &\approx \vec{v}_n + \tau \vec{a}_n \\ \vec{r}_{n+1} &\approx \vec{r}_n + \tau (\vec{v}_n + \vec{v}_{n+1}) / 2 \\ &\approx \vec{r}_n + \tau \vec{v}_n + \tau^2 \vec{a}_n / 2 \end{aligned} \right\}$$

Midpoint Method

Note that the midpoint method is exact for projectile motion without air resistance.

Local Error Versus Global Error

If we attempt to estimate the error in the methods, we need to make an estimate of the global error. For the Euler method, the error is of order τ^2 , but if we assume that this error adds, the global error (G) = the local error \times the number of steps taken (N), or $G \approx N\tau^2$

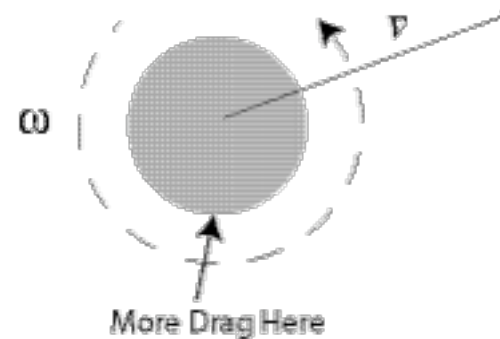
$$\approx \frac{T}{\tau} \tau^2 \approx \tau$$

The Euler method has a local error of τ^2 but a global error of τ

The Effects of Spin – Magnus Force

The force due to a spinning ball, rotating at angular velocity ω and moving a velocity v is

$$\vec{F}_m = S_o \vec{\omega} \times \vec{v}$$



The drag coefficient S_0 can be variable, for simplicity assume it is constant. Note that S_0/m is a dimensionless number.

Simple Pendulum

The Equation of motion for a simple pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

If θ is small, $\sin\theta \approx \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta$$

$$\Rightarrow \theta(t) = C_1 \cos\left(\frac{2\pi t}{T_s} + C_2\right) \quad T_s = 2\pi\sqrt{\frac{L}{g}}$$

C_1 and C_2 are constants

Simple Pendulum - Total Energy

The total energy (E) of a pendulum is simply the sum of the kinetic and potential energy

$$E = \frac{1}{2}mL^2\omega^2 - mgL\cos\theta \quad \omega = \frac{d\theta}{dt}$$

Which is conserved $E = -mgL\cos\theta_m$

Where θ_m is the maximum angle of travel of the pendulum.

Equating the above 2 equations and solving for ω gives

$$\omega^2 = \frac{2g}{L}(\cos\theta - \cos\theta_m)$$

$$\frac{dt}{d\theta} = \frac{1}{\sqrt{\frac{2g}{L}(\cos\theta - \cos\theta_m)}}$$

The pendulum's period T is 4x the time it takes to go from $\theta = \theta_m$

to $\theta=0$, so $\frac{T}{4} = \int_0^{\theta_m} \frac{d\theta}{\sqrt{\frac{2g}{L}(\cos\theta - \cos\theta_m)}} = 2\sqrt{\frac{L}{g}} \int_0^{\theta_m} \frac{d\theta}{\sqrt{\left(\sin^2 \frac{\theta_m}{2} - \sin^2 \frac{\theta}{2}\right)}}$

$$= 4\sqrt{\frac{L}{g}} K\left(\sin^2 \frac{\theta_m}{2}\right)$$

where $K(x) \equiv \int_0^{\pi/2} \frac{dz}{\sqrt{1-x^2 \sin^2 z}}$ is an elliptic integral of the first kind

Note that in MATLAB (elliptke): $K_{MATLAB}(x) \equiv \int_0^{\pi/2} \frac{dz}{\sqrt{1-x \sin^2 z}}$

Second order in time Methods

With the desire to get higher accuracy in our time-stepping methods, we will look at using central differences. One can easily show (via Taylor expansion) that

$$f'(t_n) = \frac{f(t_{n+1}) - f(t_{n-1})}{2\tau} + O(\tau^2)$$

So that the error term is of second order. In other words, if you half the timestep (τ) the error goes down by a factor of 4.

Leap Frog Method

If we define the solutions at time t_n as \vec{r}_n and \vec{v}_n

We write the second order discretized equations as

$$\frac{\vec{r}_{n+2} - \vec{r}_n}{2\tau} + O(\tau^2) = \vec{v}_{n+1} \quad \frac{\vec{v}_{n+1} - \vec{v}_{n-1}}{2\tau} + O(\tau^2) = \vec{a}(\vec{r}_n)$$

Note that the equation for position is centered at time $t+\tau$ while the equation for velocity is centered at time t . Rewriting, we get

$$\vec{v}_{n+1} = \vec{v}_{n-1} + 2\tau\vec{a}(\vec{r}_n) + O(\tau^3)$$

$$\vec{r}_{n+2} = \vec{r}_n + 2\tau\vec{v}_{n+1} + O(\tau^3)$$

The scheme has each set of variables “Leap-Frogging” each other in time. To start the scheme, one needs the position at $n=0$, for which the simplest is to use a backward Euler step.

Basic Leap Frog Algorithm

Given an initial position and velocity \vec{r}_1 and \vec{v}_1 the scheme is as follows:

1. Get an estimate of \vec{r}_0 and \vec{v}_0 using a backward Euler step:

$$\vec{r}_0 = \vec{r}_1 - \tau\vec{v}_1 \quad \text{and} \quad \vec{v}_0 = \vec{v}_1 - \tau\vec{a}_1$$

2. Compute the new position \vec{r}_2 as $\vec{r}_2 = \vec{r}_0 - 2\tau\vec{v}_1$

3. Compute the new velocity \vec{v}_2 as $\vec{v}_2 = \vec{v}_0 - 2\tau\vec{a}_1$

4. Compute the new position \vec{r}_3 as $\vec{r}_3 = \vec{r}_1 - 2\tau\vec{v}_2$

5. Repeat from 3, using the equations

$$\vec{v}_{n+1} = \vec{v}_{n-1} + 2\tau\vec{a}(\vec{r}_n) + O(\tau^3) \quad \text{and} \quad \vec{r}_{n+2} = \vec{r}_n + 2\tau\vec{v}_{n+1}$$

The Leap Frog scheme is often written as

$$\vec{v}_{n+\frac{1}{2}} = \vec{v}_{n-\frac{1}{2}} + \tau\vec{a}_n \quad \vec{r}_{n+1} = \vec{r}_{n-1} + \tau\vec{v}_{n+\frac{1}{2}}$$

Verlet Method

Another approach is to dispense with velocity in calculating the position and solve the 2nd order equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}(\vec{r}_n)$

Using the finite difference equation for second derivative, we get $\frac{\vec{r}_{n+1} - 2\vec{r}_n + \vec{r}_{n-1}}{\tau^2} + O(\tau^2) = \vec{a}_n$

or $\vec{r}_{n+1} = 2\vec{r}_n - \vec{r}_{n-1} + \tau^2\vec{a}_n + O(\tau^4)$

The velocity is computed as a post process: $\vec{v}_n = \frac{\vec{r}_{n+1} - \vec{r}_{n-1}}{2\tau} + O(\tau^2)$

And can be started as $\vec{r}_0 = \vec{r}_1 - \tau\vec{v}_1 + \frac{\tau^2}{2}\vec{a}_1$