Simple Oscillators

Some day the program director will attain the intelligent skill of the engineers who erected his towers and built the marvel he now so ineptly uses.

Lee De Forest (1873-1961)

OBJECTIVES

To observe some general properties of oscillatory systems. To demonstrate the use of an RLC circuit as a filter.

THEORY

Oscillatory systems are very common in nature and technology, where they are used to either exaggerate or attenuate some response. Automobiles are mounted on springs to attenuate sharp bumps from the roadway and provided with a damper so that the oscillations are limited in time. Both animals and musical instruments produce sound by coupling a noise source to a resonant system which picks out and efficiently radiates certain frequencies. Electronic circuits consisting of inductors and capacitors are used to select or attenuate particular frequency ranges, as in tuning a radio or rejecting noise from the AC power lines. Fortunately, all oscillatory systems have many features in common. In this experiment we will study an electrical system because it is convenient to manipulate, but the results are really quite general.

The circuits of interest are shown in Fig. 1, including sine-wave sources. We start with the series connection, writing Kirchoff's law for the loop in terms of the charge q_C on the capacitor and the current $i = dq_C/dt$ in the loop. The sum of the voltages around the loop must be zero, so

$$v_L + v_R + v_C = v_D \sin(\omega t) \tag{1}$$

We then use the relations

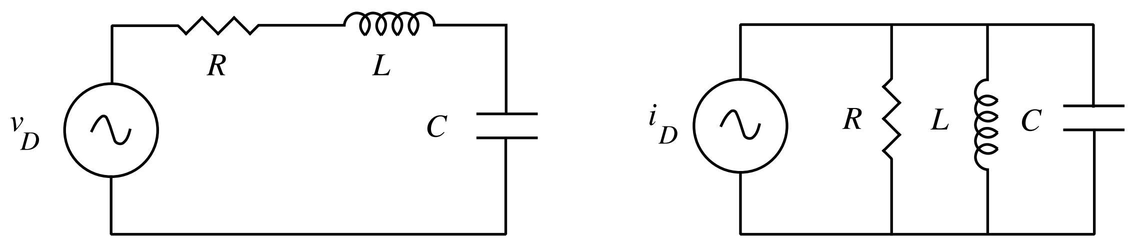

Fig. 1 Idealized series and parallel RLC circuits. The series circuit is driven by a voltage source, while the parallel circuit has a current source.

$$v_L = L\frac{di_L}{dt} \quad v_R = i_R R \quad v_C = \frac{q_C}{C} \quad i_C = \frac{dq_C}{dt} \tag{2}$$

and the fact that the current through all the elements is the same to obtain a differential equation in q_C

$$L\frac{d^2 q_C}{dt^2} + R\frac{dq_C}{dt} + \frac{q_C}{C} = v_D \sin(\omega t) \tag{3}$$

For reasons that will become clear shortly, we rewrite this as

$$\frac{d^2 q_C}{dt^2} + \frac{2}{\tau}\frac{dq_C}{dt} + \omega_0^2 q_C = \frac{v_D}{L}\sin(\omega t) \tag{4}$$

with $\tau = 2L/R$ and $\omega_0^2 = 1/LC$. The complete solution to this equation will be the sum of a solution to the homogeneous equation, with right hand side zero, and the inhomogeneous equation with non-zero right hand side. We will examine the two solutions separately.

If the resistance in the circuit is small, the homogeneous solution is of the form

$$q_C = q_{C0}\mathrm{e}^{-t/\tau}\cos(\omega_1 t + \phi) \tag{5}$$

Where q_{C0} and ϕ are determined by initial conditions, and

$$\omega_1 = \omega_0\left[1 - (\omega_0\tau)^{-2}\right]^{1/2} \tag{6}$$

This solution is plotted in Fig. 2 for a case where the capacitor is initially charged and no current is flowing. (For a mass on a spring the equivalent situation would be to pull the mass aside and release it from rest.) Evidently there are oscillations at ω_1, approximately equal to ω_0, within an exponential envelope. Note that the amplitude falls to 1/e of the initial value when $t = \tau$.

As τ gets smaller (larger resistance), ω_1 becomes smaller and finally imaginary. The corresponding solutions do not oscillate at all. For $\omega_0 < 1/\tau$, there are two exponentials

$$q_C = A_1\mathrm{e}^{-t/\tau_1} + A_2\mathrm{e}^{-t/\tau_2} \tag{7}$$

where τ_1 and τ_2 differ somewhat from τ. When $\omega_0 = 1/\tau$ the solution is slightly simpler:

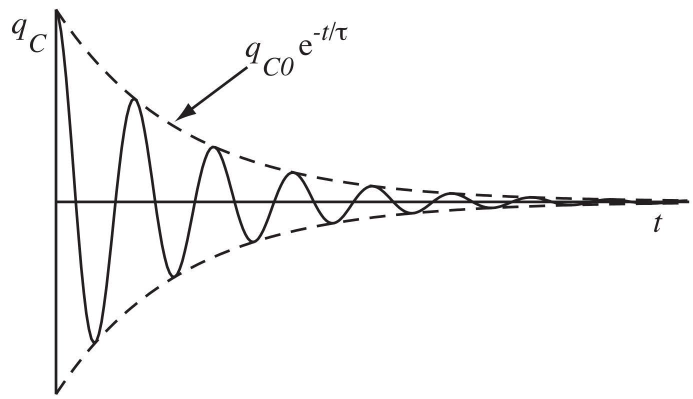

Fig. 2 Damped oscillation, showing decay envelope

$$q_C = \left(A_1 + A_2 t\right)e^{-t/\tau} \tag{8}$$

If the capacitor is initially charged, these results tell us we will get a pure decay for sufficiently large R. The case $\omega_0 = 1/\tau$ is referred to as critical damping because the charge reaches zero in the shortest time without changing polarity.

The inhomogeneous solution describes a steady oscillation at the frequency of the driving voltage:

$$q_C = A\sin(\omega t + \phi) \tag{9}$$

We can find A and ϕ by substituting into the differential equation and solving:

$$A = \frac{v_D / L}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(2\omega/\tau\right)^2\right]^{1/2}} \tag{10}$$

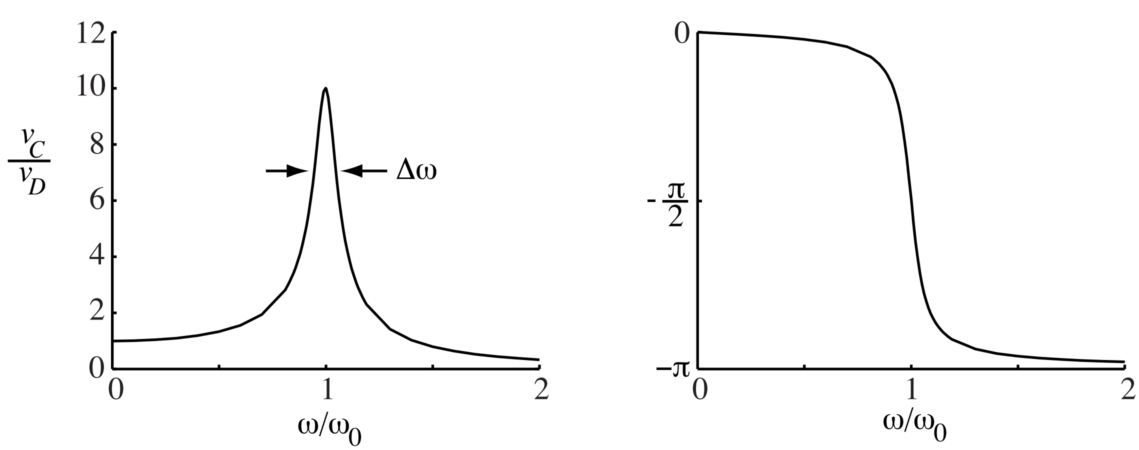

Fig. 3 Amplitude and phase of capacitor voltage as a function of frequency.

$$\tan\phi = \frac{2\omega/\tau}{\left(\omega^2 - \omega_0^2\right)} \tag{11}$$

These two equations are plotted in Fig. 3, where we use the fact that $v_C = q_C/C$ to plot the voltage across the capacitor relative to the driving voltage.

The angular frequency for maximum amplitude is given by

$$\omega_{peak} = \omega_0\left[1 - 2/\left(\omega_0\tau\right)^2\right]^{1/2} \tag{12}$$

At the maximum, the oscillation amplitude is considerably greater than the driving amplitude. In fact, if τ were infinite (no damping), the response would be infinite at ω_0. The phase shift of v_C, relative to the driving voltage, is also shown in Fig. 3. It changes smoothly from zero at low frequency to -180° well above the resonance, going through -90° precisely at ω_0.

Since the shape of the peak in v_C characterizes the resonance, it is convenient to have some parameter to specify the sharpness of the peak. Traditionally, this is taken to be the full width $\Delta\omega$, shown in Fig. 3, at which the voltage or current have fallen to $1/\sqrt{2}$ of their peak value. The reason for this choice is that the power dissipation is proportional to i^2, so these frequencies correspond to the points at which the power dissipation is half of the maximum. The width $\Delta\omega$ is not usually specified directly, but rather a normalized quality factor $Q = \omega_0/\Delta\omega$, is used. A circuit with a sharp resonance is then referred to as a high-Q circuit. Using Eq. 10 and the definitions of τ and ω_0, we can express Q in terms of various measureable quantities

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0\tau}{2} = \frac{\omega_0 L}{R} = \frac{v_C}{v_D} \tag{13}$$

demonstrating the intimate relation between time and frequency response parameters. (In obtaining this result, we assumed $\omega_0\tau \gg 1$, so that $\omega \approx \omega_0$ at the peak.)

The analysis of the parallel circuit proceeds along an analogous path, except that it is easier to consider currents rather than voltages. Charge conservation requires that

$$i = i_R + i_L + i_C \tag{14}$$

where $i = i_D\sin(\omega t)$ is the current from the source. To rewrite this as a differential equation in q_C we use the relations

$$i_R = \frac{v_R}{R} \quad i_C = \frac{dq_C}{dt} \quad v_C = \frac{q_C}{C} \quad v_L = L\frac{di_L}{dt} \tag{15}$$

and the fact that the voltages across the elements are all the same. This leads to

$$\frac{di}{dt} = \frac{d^2q_c}{dt^2} + \frac{2}{\tau'}\frac{dq_c}{dt} + \omega_0^2 q_C \tag{16}$$

where $\tau' = 2RC$. We immediately notice that the homogeneous equation is identical to that for the series circuit, except for a different time constant. The undriven oscillations are, therefore, qualitatively the same as before.

To find the driven response to a current of constant amplitude i_D we again assume a solution as in Eq. 9 and substitute into the differential equation. After some algebra, we get

$$A = \frac{i_D\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(2\omega/\tau'\right)^2\right]^{1/2}} \tag{17}$$

As expected, the maximum voltage amplitude occurs when the driving frequency is near ω_0. Conversely, if we were to drive the circuit with a constant voltage, we would find the minimum current amplitude at resonance, indicating that the reactance of the parallel circuit is maximum at resonance.

EXPERIMENTAL PROCEDURE

A realistic version of the series RLC circuit is shown in Fig. 4. The oscilloscope connections allow us to display the voltage across the capacitor, which is proportional to q_C, and

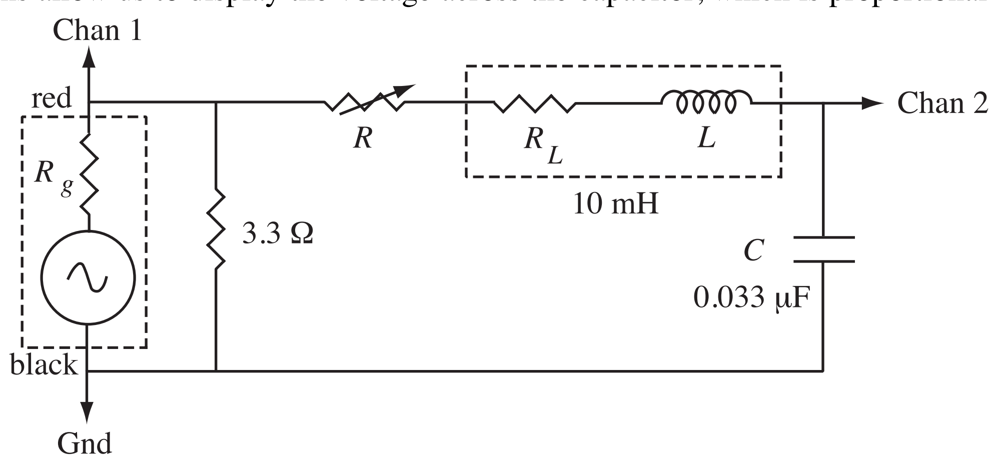

Fig. 4 Circuit used for series RLC measurements. Note the 3.3Ω resistor used to reduce the damping effect of the function generator.

the driving voltage. Our function generator is equivalent to an ideal voltage source in series with a 50 Ω resistor, labeled R_g in the diagram. Since R_g is large enough to affect the damping of the circuit, we add the 3.3 Ω resistor in parallel to reduce the effective resistance of the generator. The other resistance, R_L, represents the resistance of the wire in the inductor and some energy losses associated with the other components.

Free oscillations

To study the homogeneous solution we will use the square-wave output of the function generator to charge the capacitor and then allow it to discharge. (The generator output actually goes from $+v$ to $-v$, rather than switching from v to zero, but this makes no material difference.)

Wire the circuit as shown, using the 20 kΩ variable resistor for R. Set up the scope to trigger on the function generator signal and to display v_C. With the variable resistor at minimum resistance and the function generator set for square waves at about 150 Hz, you should get a series of plots like Fig. 2. Each decay should be perched on a part of the square wave and show a flat region where the oscillations have completely died away. The decay that you observe is due to the total resistance of R_L in series with the parallel combination of R_g and the 3.3 Ω resistor.

Now look at the effect on the oscillations as you vary the resistance from zero to the maximum. Expand the display so that you can see one decay pattern clearly, and then sketch one example each of underdamped, critically damped, and overdamped decays. Be sure to clearly indicate the time scale, so that one can see how long it takes for the voltage to reach zero.

We can actually measure τ by finding the time at which the signal height falls to 1/e of its original height. Set the variable resistor to R = 0 or, better, replace it with a wire to insure minimum resistance between the function generator and inductor. Adjust the function generator amplitude and the scope controls so that the baseline of the decay curve is at the bottom of the screen, and the start of the first oscillation is exactly at the top. Expand the time scale to get a plot that looks like Fig. 5. Since the initial amplitude is 8 divisions, the 1/e point is at 3 divisions.

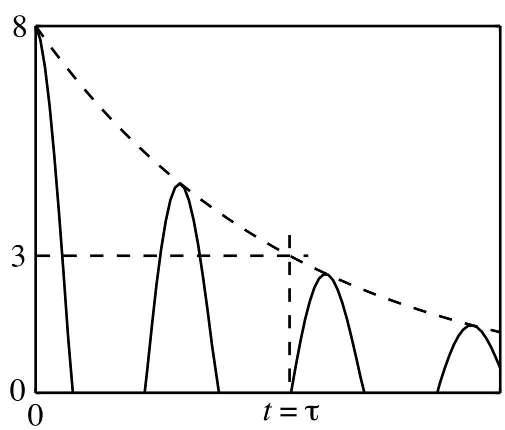

Fig. 5 Scope display used for measurement of τ

You will need to interpolate between the peaks to estimate the time at which the envelope of the oscillations reaches 3 divisions, as indicated in the figure. Be sure that the scope time scale control is in the calibrated position.

Driven oscillations

To study the inhomogeneous solution we only need to switch the function generator to sine-wave output, keeping the variable resistor at zero. By varying the frequency of the function generator you should easily find the resonance. Check the phase shift on resonance and at frequencies well above and below resonance. Does it change as suggested by Fig. 3?

Next, you should sketch v_C as a function of frequency, clearly showing peak amplitude, center frequency and peak width. You may have noticed that v_D, shown on channel 1, decreases slightly near resonance. This occurs because the current in the series circuit is maximum at resonance and the voltage drop across R_g is significant. Fortunately the output of the ideal voltage source inside the dashed lines of Fig. 4 is approximately independent of frequency. (How could you check this assertion?) Therefore, you do not need to adjust the output amplitude as you change frequency. Does your sketch look like Fig. 3?

You can get an accurate measure of the width by finding the frequencies just above and just below the resonance at which v_C is reduced by $1/\sqrt{2}$ from the peak value. The width $\Delta\omega$ is then $2\pi\Delta f$, where the 2π converts from frequency in Hertz to angular frequency.

Using Eq. 13, compare the Q of your circuit as determined from the width of the resonance curve, from the ratio of maximum v_C to v_D and from your previous measurement of τ. How do the various values compare?

Parallel resonance and radio receiver

Rather than repeat these measurements for the parallel circuit, we will proceed directly to a simple AM radio, as shown in Fig. 6. In this circuit the parallel resonant circuit is used as a

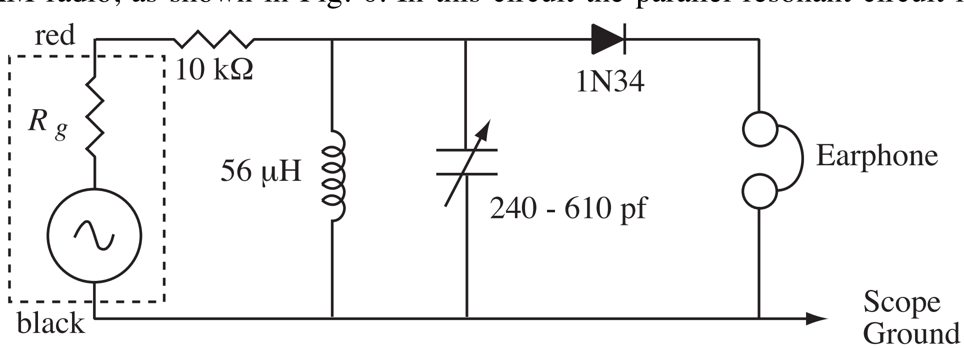

Fig. 6 Radio circuit, shown with a radio-frequency signal generator and resistor to simulate an antenna. The symbol with two circles connected by a semicircle stands for an earphone.

filter to select one frequency from the many that can excite the antenna. The variable capacitor is adjusted so that maximum response occurs at the desired frequency. A diode detects the amplitude modulation and the earphone allows us to hear the resulting audio signal.

It might seem attractive to use a series-resonant circuit to get some voltage magnification, but this is not practical. At the frequencies used for AM broadcasting, 550 kHz to 1.6 MHz in the US, a convenient antenna is much less than a wavelength long. Although not obvious, an antenna which is short in this sense has a high effective resistance. You have already demonstrated that putting a high resistance into a series circuit damps the response and destroys the Q, so it would not be useful. By connecting the parallel resonant circuit in series with the high antenna impedance we can effectively create a voltage divider with one frequency-sensitive element. On resonance, the voltage across the LC pair is almost equal to the source voltage, while off resonance the voltage is much lower because of the low impedance of the LC circuit.

An RF signal generator is used to simulate an AM radio station. To understand the signal, connect channel 1 of the scope to the RF OUTPUT, and channel 2 to the modulation OUTPUT/INPUT. Flip the modulation switch to EXT to turn off modulation. Setting the scope to view and trigger on channel 1, you should see a sinusoidal wave at a frequency of a megahertz or so, depending on the setting of the FREQUENCY RANGE and main dial. Verify that this high frequency changes as you change the dials. The steady sine wave at an assigned frequency is called the carrier for AM radio.

Program material is transmitted by varying the amplitude of the carrier, a process called amplitude modulation. Set the scope to view and trigger on channel 2, and set the modulation switch to INT (1KHz). You should see a poor-quality sine wave at about 1 kHz. Now display channel 1 on the scope, leaving the trigger on channel 2. If you slow the sweep enough you should see that the amplitude of the high frequency carrier varies regularly at the much lower modulation frequency. The modulation amplitude can be changed by adjusting the MODULATION LEVEL control. This shows how audio frequency information can be encoded onto a higher frequency signal for transmission.

To start the measurements, connect the RF signal generator to the 10 kΩ resistor, variable capacitor and inductor as shown in Fig. 6, omitting the diode and earphone for now. Connect channel 1 of the scope to the capacitor, and leave channel 2 on the modulation output of the RF generator. Set the generator for about 1 MHz, and vary the capacitor setting to find a peak in the voltage across the capacitor. Record the capacitor settings at the peaks for a few frequencies. Does your circuit seem to resonate in the AM radio band? For later reference sketch the waveform of the voltage across the capacitor with modulation present. Be sure to clearly indicate the zero-voltage level.

Continue the receiver construction by adding the diode to the circuit, but use a 1 kΩ resistor in place of the earphone. Use channel 1 of the scope to sketch the modulated voltage waveform appearing across the resistor, clearly indicating the zero level. Describe the effect of the diode on the signal. Finally, complete the receiver by replacing the 1 kΩ resistor with the headphone. This time, sketch the voltage waveform across the earphone, and indicate the zero-voltage level. What happened to the high frequency part? (Hint: The earphone acts like a capacitor. What is the reactance of a capacitor at very high frequency?) Why is the time-average value of the low-frequency signal not zero? Can you hear the modulation through the earphone?

When you have the circuit working properly, connect the antenna (the long wire attached to one of the binding-post terminals on your plugboard) in place of the RF generator and 10 kΩ resistor. Be sure the indicated ground point is connected to the ground terminal on the scope or RF generator. Can you hear some radio stations? Record the capacitor settings for whatever stations you can find.