Last Name: First Name:

Physics 102 Spring 2003: Test 1—Free Response and Instructions

- **Print your LAST and FIRST name on the front of your blue book, on this question sheet, the multiple-choice question sheet and the multiple-choice answer sheet.**
- **TIME ALLOWED 90 MINUTES**
- **The test consists of two free-response questions and ten multiple-choice questions.**
- **The test is graded on a scale of 100 points; the first free-response question accounts for 40 points, the second free-response question accounts for 30 points, and the multiple-choice questions account for 30 points.**
- **Answer the two free-response questions in your blue book. Answer the multiple-choice questions by marking a dark X in the appropriate column and row in the table on the multiple-choice answer sheet.**
- **Consult no books or notes of any kind. You may use a hand-held calculator in non-graphing, non-programmed mode.**
- **Do NOT take test materials outside of the class at any time. Return this question sheet along with your blue book and multiple-choice question sheet.**
- **Write and sign the Pledge on the front of your blue book.**

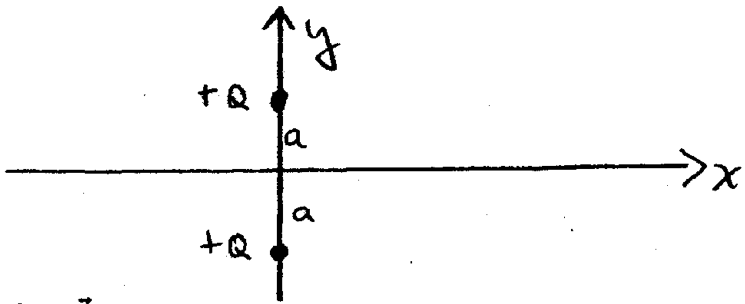
Show your work for the free-response problems, including neat and clearly labelled figures, in your blue book. Answers without explanation (even correct answers) will not be given credit.

I. (40 pts) A model of the electron treats it as a sphere of radius R_e with the charge $-e$ uniformly distributed throughout its volume. The coordinate r measures the distance from the center of the sphere. Express your answers in terms of e, R_e, ϵ_0, and r.

(a) Determine the electric field $\vec{E}(r)$ for all values of r.
(b) Taking the zero of the electrostatic potential to be at infinity, determine $V(r)$ for all values of r.
(c) Sketch $V(r)$ for all r.
(d) Determine the energy density in the electric field $u_E(r)$ for all values of r.
(e) Determine the total energy stored in the electric field by integrating the energy density $u_E(r)$ over all space.
(f) A classical model of the electron equates the rest mass energy of the electron $m_e c^2$ to the total energy stored in the electric field. From your result in (e) determine this so-called classical radius of the electron R_e in terms of the charge of the electron e, the rest mass energy of the electron $m_e c^2$, and other physical constants.

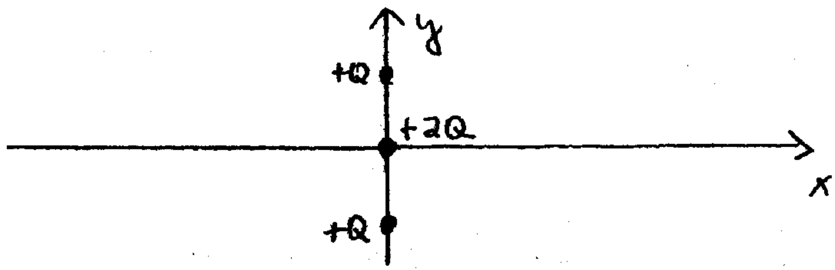
We know from direct measurements of the electron size in scattering experiments that in fact the electron is much smaller than this classical model would indicate. So, in fact, the rest mass energy of the electron must have some other origin than the energy stored in the electric field.

II. (30 pts) Two positive point charges $+Q$ are located on the y-axis at $y = \pm a$ as shown below.



(a) Determine the electric field $\vec{E}$ on the x-axis, for all values of x.
(b) Determine the electrostatic potential $V(x)$ for all points on the x-axis.
(c) Sketch $V(x)$ for all values of x on the x-axis.
(d) From your plot in (c), determine at which points on the x-axis the electric field is zero.

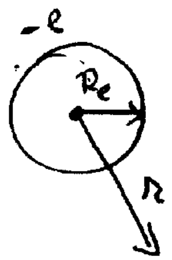
Now suppose a third positive charge of magnitude $+2Q$ is brought in from infinity and placed at the origin.



(e) How much work must be done by an external agent to put the $+2Q$ charge at the origin?
(f) What is the total electrostatic energy of this three-charge configuration?
(g) If all three charges are released and allowed to move freely, what is the sum of their kinetic energies when all the particles are very far from each other?

Phys 102
Exam 1 - Spring 2003

I.



(a) For $r > R_e$ $\vec{E}$ is the same as for a point charge, by Gauss' Law & symmetry:

$$\boxed{\vec{E} = \frac{-e}{4\pi\varepsilon_0 r^2}\hat{r} \quad r > R_e}$$

radially <u>inward</u>

For $r < R_e$, we can determine $\vec{E}$ from Gauss' Law. Take

$\rho = \dfrac{-3e}{4\pi R_e^3}$ $\qquad \oint \vec{E}\cdot d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_0} = \int \dfrac{\rho}{\varepsilon_0}\,dV$

$$4\pi r^2 E_r = \frac{4\rho\pi r^3}{3\varepsilon_0}$$

$$E_r = \frac{\rho r}{3\varepsilon_0} = -\frac{3er}{4\pi R_e^3 \cdot 3\varepsilon_0}$$

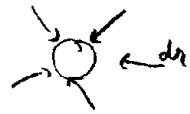
$$\boxed{\vec{E} = \frac{-er}{4\pi\varepsilon_0 R_e^3}\hat{r} \quad r < R_e}$$

(b) For $r > R_e$, the potential is the same as for a point charge, $V(r) = -\int_r^\infty \vec{E}\cdot d\vec{r} = \frac{e}{4\pi\varepsilon_0}\int \frac{dr}{r^2} = \frac{-e}{4\pi\varepsilon_0 r}\Big|_r^\infty$

$$\boxed{V(r) = \frac{-e}{4\pi\varepsilon_0 r} \quad r > R_e}$$

We can determine $V(r)$ for $r < R_e$ by finding ΔV from $R_e \to r$

$$\Delta V = -\int_{R_e}^{r} \vec{E} \cdot d\vec{r}'$$



$\vec{E} \parallel d\vec{r}$, so $\vec{E} \cdot d\vec{r} = E\,dr$

$$\Delta V = \frac{e}{4\pi\varepsilon_0 R_e^3} \int_{R_e}^{r} r'\,dr' = \frac{e}{4\pi\varepsilon_0 R_e^3} \left(\frac{r'^2}{2} \right)\Bigg|_{r'=R_e}^{r} = \frac{e\,r^2}{8\pi\varepsilon_0 R_e^3} - \frac{e\,R_e^2}{8\pi\varepsilon_0 R_e^3}$$

$$\Delta V = \frac{e}{8\pi\varepsilon_0} \left(\frac{r^2}{R_e^3} - \frac{1}{R_e} \right)$$

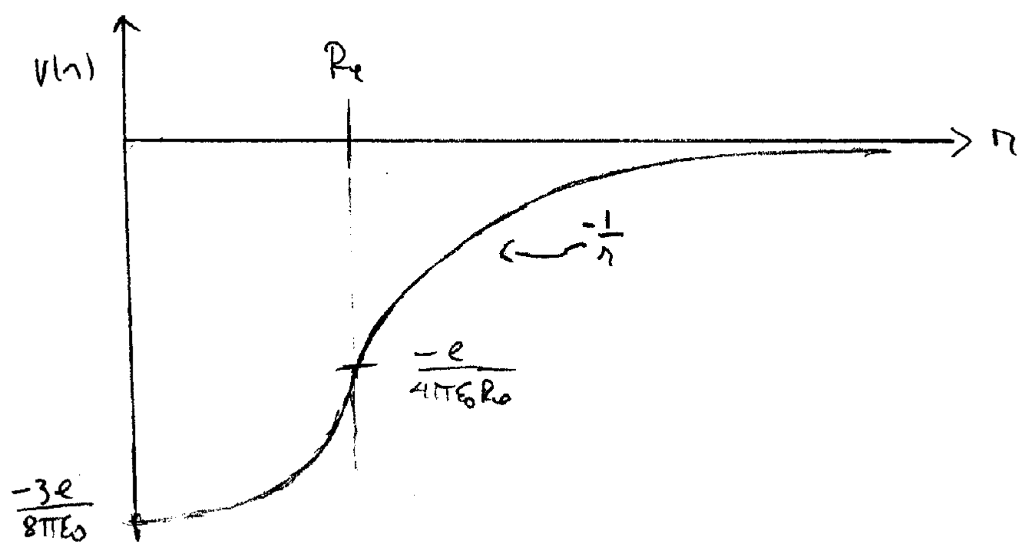
Note on limits & the sign of ΔV: for $r = R_e$, ΔV must $= 0$, since we started at R_e. That is, $\Delta V = 0$ when $\Delta r = 0$. The expression for ΔV above satisfies that. As we go toward $r \to 0$, V must be decreasing, since we are moving in the same direction as $\vec{E}$ ($\vec{E} \parallel d\vec{r}$). This expression also has this behavior.

Now we must add ΔV to $V(r = R_e)$

$$\underset{r<R_e}{V(r)} = V(R_e) + \Delta V = \frac{-e}{4\pi\varepsilon_0 R_e} + \frac{e}{8\pi\varepsilon_0} \left(\frac{r^2}{R_e^3} - \frac{1}{R_e} \right)$$

$$\boxed{\underset{r<R_e}{V(r)} = \frac{e}{8\pi\varepsilon_0} \left(-\frac{3}{R_e} + \frac{r^2}{R_e^3} \right)}$$

(c)



(d) $u_E(r) = \dfrac{\varepsilon_0 E^2}{2}$

$$\boxed{\text{For } r > R_e, \quad u_E(r) = \frac{e^2}{32\pi^2 \varepsilon_0 r^4}}$$

$$\boxed{\text{For } r < R_e, \quad u_E(r) = \frac{e^2 r^2}{32\pi^2 \varepsilon_0 R_e^6}}$$

(e) Integrate $u_E(r)$ in two pieces, U_1 $(r > R_e)$ and U_2 $(r < R_e)$

$r > R_e$, $\quad U_1 = \displaystyle\int_{r=R_e}^{\infty} u_E(r)\, dV \qquad dV = 4\pi r^2 dr$

$$U_1 = \frac{e^2}{32\pi^2 \varepsilon_0} \cdot 4\pi \int_{R_e}^{\infty} \frac{r^2 dr}{r^4} = \frac{e^2}{8\pi\varepsilon_0} \int_{R_e}^{\infty} \frac{dr}{r^2} = \frac{e^2}{8\pi\varepsilon_0} \left(-\frac{1}{r}\right)\Bigg|_{R_e}^{\infty}$$

$$U_1 = \frac{e^2}{8\pi \varepsilon_0 R_e}$$

for $r < R_e$, $\quad U_2 = \displaystyle\int_0^{R_e} u_E(r)\, dV = \frac{e^2 (4\pi)}{32\pi^2 \varepsilon_0 R_e^6} \int_0^{R_e} r^4 dr = \frac{e^2}{8\pi\varepsilon_0 R_e^6} \cdot \frac{R_e^5}{5}$

$$U_2 = \frac{e^2}{40\pi\varepsilon_0 R_e}$$

$$U_{tot} = U_1 + U_2 = \frac{e^2}{40\pi\varepsilon_0 R_e}\left[1 + 5\right] = \frac{3e^2}{20\pi\varepsilon_0 R_e}$$

$$\boxed{U_{tot} = \frac{3e^2}{20\pi \varepsilon_0 R_e}}$$

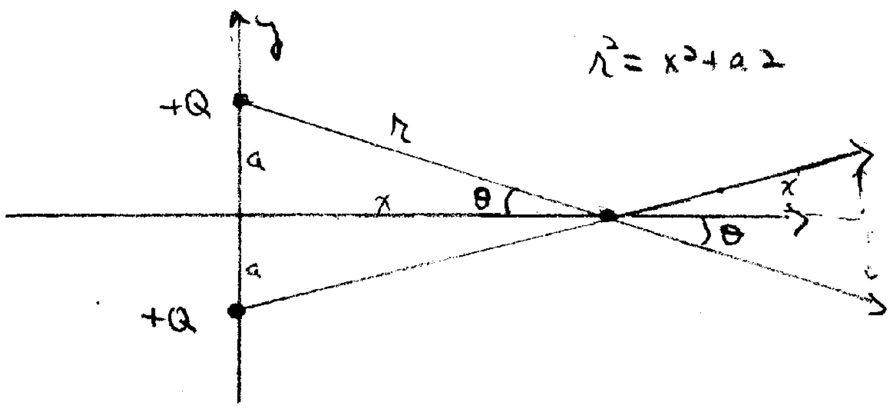
(f) Set $U_{TOT} = m_e c^2$ (i.e, energy in electric field = rest mass energy)

$$\frac{3e^2}{20\pi \epsilon_0 R_e} = m_e c^2$$

$$\boxed{R_e = \frac{3e^2}{2\pi \epsilon_0 m_e c^2}}$$

Note that numerically, $R_e \cong 2 \times 10^{-15}$ m, which is much bigger than the experimental limit $R_e < 10^{-20}$ m

II.



(a) $\vec{E}$ on the x-axis will have only an x-component from symmetry ⟹ y-components cancel.

$$\vec{E}(x) = E_x = \frac{2kQ}{r^2}\cos\theta\,\hat{\imath} \qquad \cos\theta = \frac{x}{r}$$

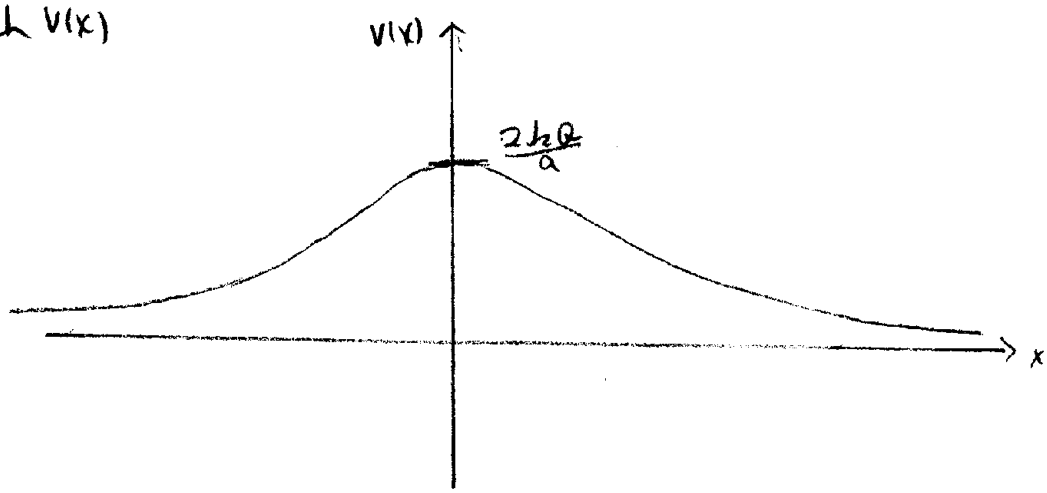
$$\boxed{\vec{E}(x) = E_x = \frac{2kQx}{(x^2+a^2)^{3/2}}\,\hat{\imath} = \frac{2Qx}{4\pi\varepsilon_0(x^2+a^2)^{3/2}}\,\hat{\imath}}$$

For $x<0$, $\vec{E}$ changes sign, but our expression also changes sign, so it is correct for all values of x.

(b) $V(x)$ is just the scalar sum of each contribution

$$\boxed{V(x) = \frac{2kQ}{r} = \frac{2kQ}{\sqrt{x^2+a^2}} = \frac{2Q}{4\pi\varepsilon_0(x^2+a^2)^{1/2}}}$$

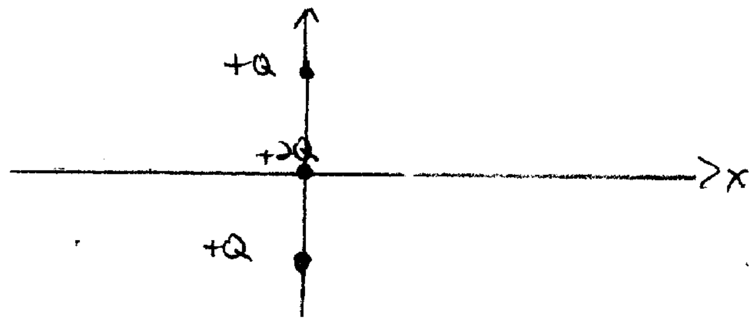
(c) Sketch $V(x)$



(d) $\vec{E}_x = 0$ where $\frac{dV}{dx} = 0$, which occurs at $x = 0$ and $x = \pm\infty$

$$\boxed{\vec{E} = 0 \text{ on the x-axis at } x = 0 \ \& \ x = \pm\infty}$$

(e)



$\Delta \text{work} = q\,\Delta V$ with $q = +2Q$

$V(x)$ at the origin is $V = \frac{2kQ}{a}$

V at $\infty = 0$, so $\Delta V = \frac{2kQ}{a}$

Therefore, the work done by an external agent to place $+2Q$ at the origin is

$$\text{Work} = \frac{2kQ}{a} \cdot 2Q = \frac{4kQ^2}{a}$$

$$\boxed{\text{Work} = \frac{4kQ^2}{a} = \frac{Q^2}{\pi \varepsilon_0 a}}$$

(f) $U_{TOT} = U_{12} + U_{13} + U_{23}$

$$= \frac{kQ^2}{2a} + \frac{2kQ^2}{a} + \frac{2kQ^2}{a} = \boxed{\frac{9kQ^2}{2a} = U_{TOT}}$$

(g) If the particles are released, all of the potential energy in (f) will be converted to kinetic energy,

$$\boxed{KE = \frac{9kQ^2}{2a}}$$

How the KE is split between the three particles will depend on details of how they are released, but the sum is always the same.