Last Name: First Name:

Physics 102 Spring 2002: Final Exam, May 6, 2002
Free Response and Instructions

- **Print your LAST and FIRST name on the front of your blue book, on this question sheet, the multiple-choice question sheet and the multiple-choice answer sheet.**
- **TIME ALLOWED 3 HOURS**
- **The test consists of four free-response questions and 20 multiple-choice questions.**
- **The test is graded on a scale of 200 points; the free-response questions account for 120 points, and the multiple-choice questions account for 80 points.**
- **Answer the four free-response questions in your blue book. Answer the multiple-choice questions by marking a dark X in the appropriate column and row in the table on the multiple-choice answer sheet.**
- **Consult no books or notes of any kind. You may use a hand-held calculator in non-graphing, non-programmed mode.**
- **Do NOT take test materials outside of the class at any time. Return this question sheet along with your blue book and multiple-choice question sheet.**
- **Write and sign the Pledge on the front of your blue book.**

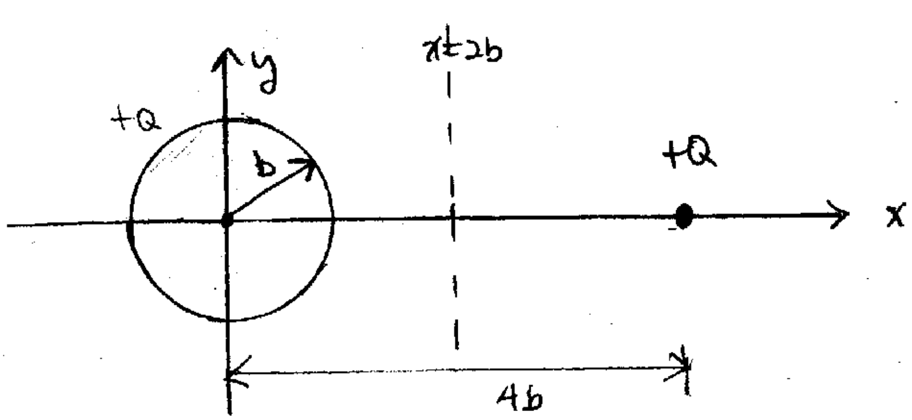
Show your work **for the free-response problems, including neat and clearly labelled figures, in your blue book. Answers without explanation (even correct answers) will not be given credit.**

I. (40 pts) A thin, insulating spherical shell of radius b is centered at the origin. It carries a total charge $+Q$, uniformly distributed over its surface.

(a) Using Gauss' law and symmetry arguments, derive, giving complete details of the derivation, the electric field everywhere in space. Sketch the magnitude of the field, $E(r)$, for all r.
(b) Taking the zero of the electrostatic potential to be at infinity, determine the electrostatic potential $V(r)$ for all r. Sketch $V(r)$ for all r.

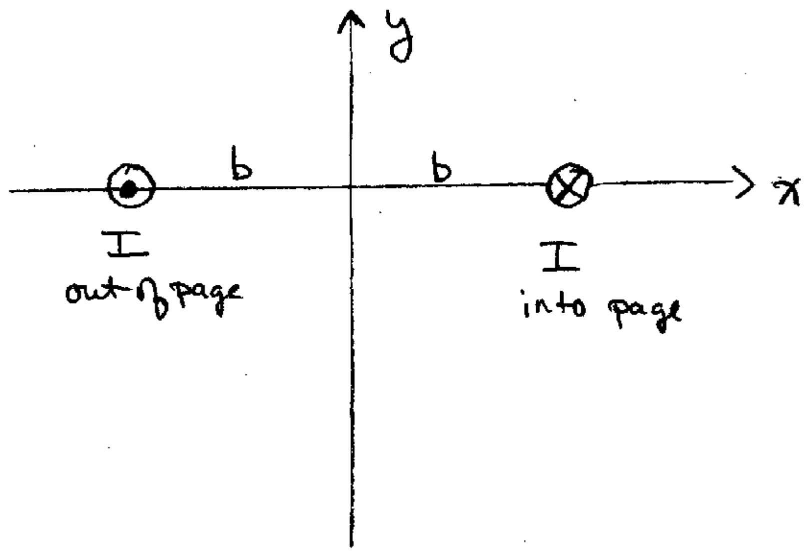
Now a second charge, a point particle of charge $+Q$ is placed on the x-axis at $x = 4b$, as shown below. The presence of this charge does not affect the distribution of charge on the spherical shell, which is still located at the origin.

(c) Determine the total electric field $\vec{E}(y)$ as a function of y on the vertical line $x = 2b$. Don't forget to include both positive and negative values for y.
(d) Determine the total electric field $\vec{E}(x)$ on the x-axis for $x > b$. Are there any points other than $+\infty$ at which the electric field is 0?
(e) Taking the zero of the potential to be at infinity, determine the total electrostatic potential $V(x)$ for all points on the x-axis for $x > b$.
(f) Holding the spherical shell and point charge of $+Q$ fixed in place, a point charge $+q$ is released from rest at the point $x = 2b, y = 2b$. Determine its kinetic energy after it has escaped to infinity.



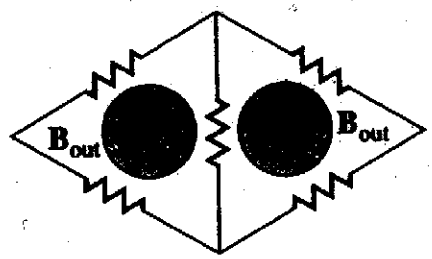
II. (25 pts) Two very long current-carrying wires, are situated as shown below. One wire is located at $x = -b$, and carries current I out of the page. The other is located at $x = +b$ and carries current I into the page.

(a) Determine the force per unit length that the wire at $x = +b$ exerts on the wire at $x = -b$.
(b)Determine the magnetic field $\vec{B}(y)$ produced by the wires on the vertical line $x = 0$, that is the line midway between them. Don't forget to consider both positive and negative values for y.
(c) A charged particle of mass m and negative charge $-q$ is released from the point $x = 0, y = b$ with velocity $\vec{v} = v_0\hat{i}$. Determine its acceleration immediately after it is released.



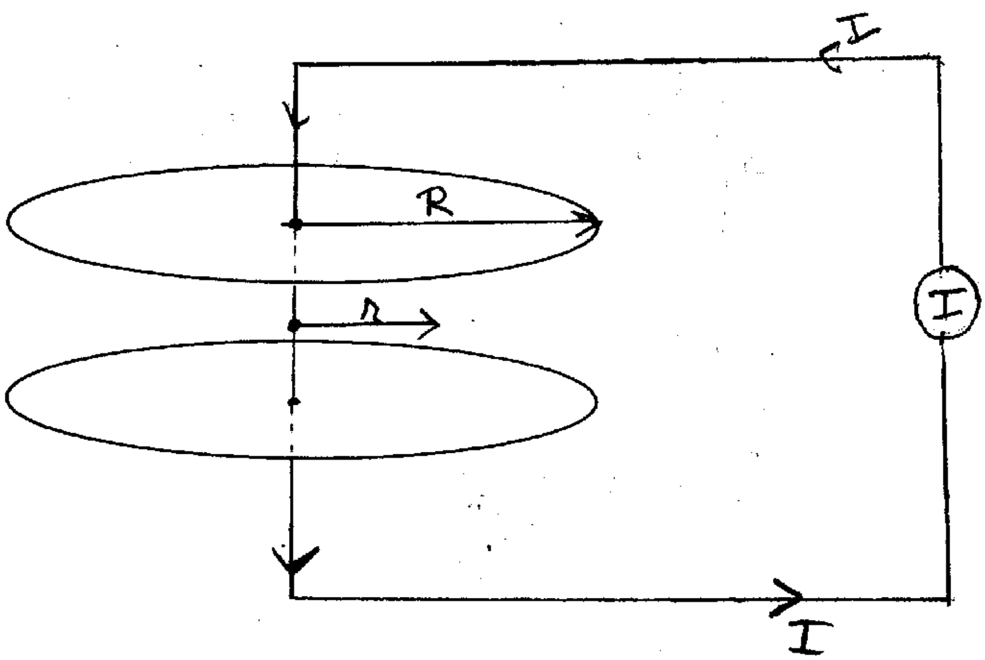
III. (25 pts) Five wires of equal length are connected to form two equilateral triangles, as shown below. Each of the five resistors has value 20Ω. Two solenoids, each 10cm in diameter, extend a long way both into and out of the page. The magnetic fields of both solenoids point out of the page. The field strength in the left-hand solenoid is increasing at 60mT/sec, and that in the right-hand solenoid is decreasing at 25mT/sec.

(a) Determine the emf $\mathcal{E}_{\mathcal{L}}$ induced in the left hand triangular loop.
(b) Determine the emf $\mathcal{E}_{\mathcal{R}}$ induced in the right hand triangular loop.
(c) Determine the current in the central wire, shared by the two loops. Which way does the current flow?



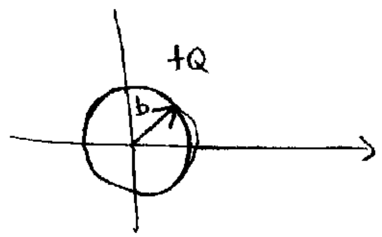
IV. (30 pts) A very large cylindrical capacitor of radius R and plate separation d is being charged slowly with constant current I. As the capacitor charges, the electric field between the plates increases with time. Take r to be the radial distance from the axis of the capacitor, as shown in the figure below.

(a) At a particular point in time, the surface charge density on the plates is σ. What is the electric field $\vec{E}$ between the plates at that time?
(b) Determine the time rate of change of the electric field between the plates, $\frac{dE}{dt}$, in terms of I, R, and other constants.
(c) Near the center of the plates, the electric field is constant in space. Determine the magnetic field $\vec{B(r)}$ between the plates for $r < R$ in terms of I, R and other constants. Be sure to indicate both direction and magnitude of the field.
(d) Neglecting fringe effects around the edge of the capacitor plates, determine the magnetic field $\vec{B(r)}$ for $r = R$ and for $r > R$.
(e) Sketch $B(r)$ for all r.



Physics 102
Final Exam

I.



(a) Gauss' Law gives

$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

Take a spherical Gaussian surface with $r > b$.
Because of the spherical symmetry, we know that $|E|$ must be the same everywhere on the Gaussian surface. We also know, from Coulomb's Law and from the spherical symmetry that $\vec{E}$ must be radially outward (the only unique direction defined by the spherical symmetry is radially outward).
Then $\vec{E} \parallel d\vec{A}$ and $\vec{E} \cdot d\vec{A} = E_r dA$

$$\int \vec{E} \cdot d\vec{A} = \int E_r dA = E_r \int dA$$

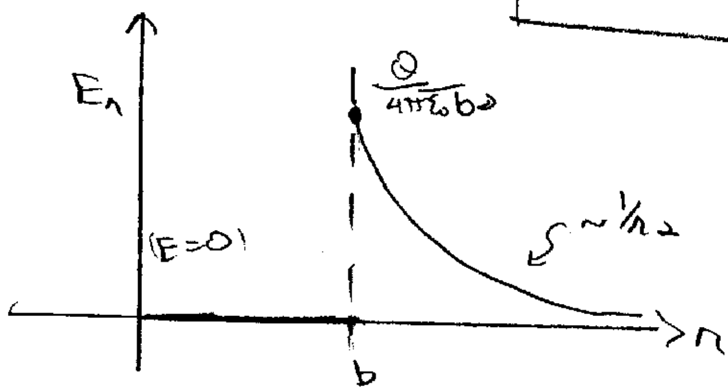
We can take E out of the integral because it is ~~constant~~ over the surface. Then Gauss' Law becomes

$$E_r \oint dA = 4\pi r^2 E_r = \frac{Q_{encl}}{\varepsilon_0}$$

for $r > b$, $Q_{encl} = +Q$ and we have

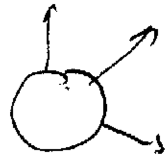
$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \quad \text{or} \quad \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad r > b}$$

for $r < b$, $Q_{encl} = 0$ and $\boxed{\vec{E} = 0 \qquad r > b}$



(b) from the definition of V,

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$



Calculate ΔV from $\infty \to r$, $\vec{E}\cdot d\vec{l} = \vec{E}\cdot d\vec{r} = -E_r dr \quad (r > b)$

$$\Delta V = \int_r^\infty E_r dr = \frac{Q}{4\pi\varepsilon_0}\int_r^\infty \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)_r^\infty = \frac{Q}{4\pi\varepsilon_0 r}$$

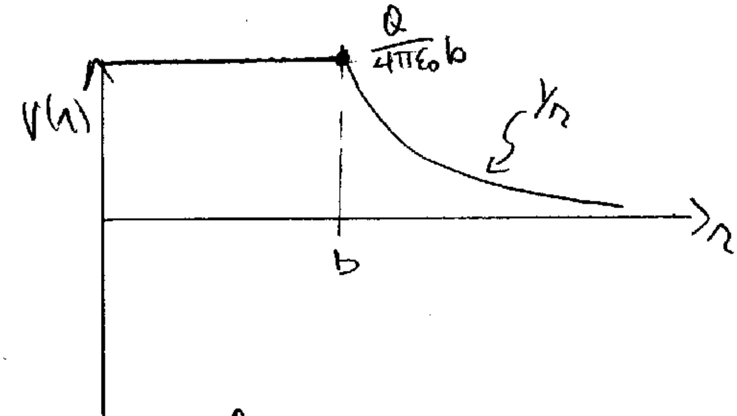
$$\Delta V = V_f - V_i = V(r) - \underbrace{V(\infty)}_{0} = \frac{Q}{4\pi\varepsilon_0 r}$$

$$\boxed{V(r) = \frac{Q}{4\pi\varepsilon_0 r} \qquad r > b}$$

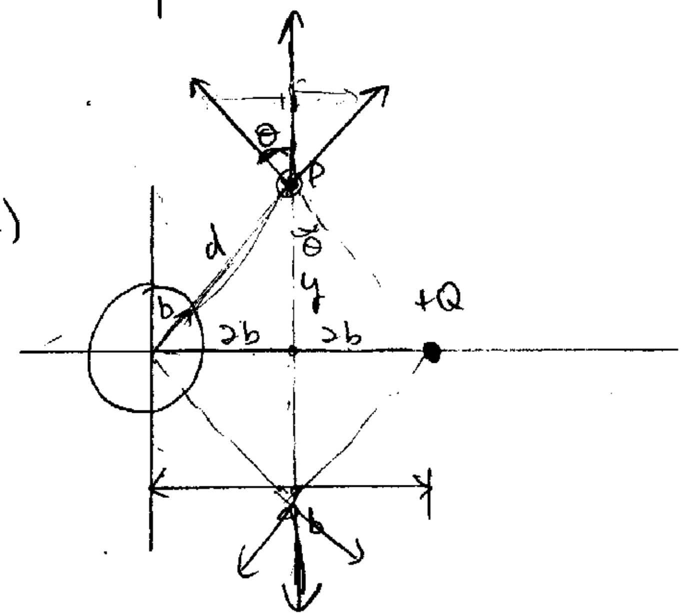
At $r = b$, E_r drops to $\emptyset$, so ΔV from $b \to 0 = 0$.

V is constant over $0 < r < b$ & equal to the value at $r = b$

$$\boxed{V(r < b) = \frac{Q}{4\pi\varepsilon_0 b} \qquad r \leq b}$$



(C)



x-components cancel

y-components add

$|E| = \frac{Q}{4\pi\varepsilon_0 d^2}$ for each charge

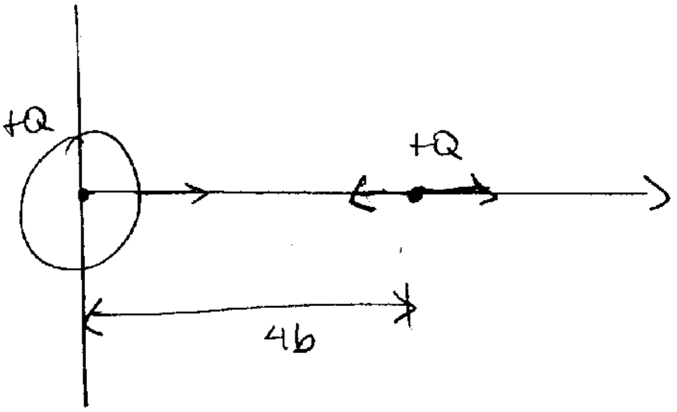
$y>0 \cdot \vec{E}(y) = \frac{2Q\cos\theta}{4\pi\varepsilon_0 d^2}\hat{j}$ $\quad \cos\theta = \frac{y}{d}$ $\quad d = \sqrt{y^2+4b^2}$

$$\boxed{\vec{E}(y) = \frac{2Qy}{4\pi\varepsilon_0 (y^2+4b^2)^{3/2}}\hat{j} \qquad y>0}$$

for $y<0$, the magnitude is the same, but direction is $-y$

$$\boxed{y<0 \quad \vec{E}(y) = \frac{-2Q|y|}{4\pi\varepsilon_0 (y^2+4b^2)^{3/2}}\hat{j}}$$

(d)



first consider $x > 4b$. $\vec{E} = E_x$ only

$$x > 4b \quad \vec{E} = \frac{Q}{4\pi\varepsilon_0 x^2}\hat{\imath} + \frac{Q}{4\pi\varepsilon_0 (x-4b)^2}\hat{\imath} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{x^2} + \frac{1}{(x-4b)^2}\right]\hat{\imath}$$

$$\boxed{\vec{E}(x) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{x^2} + \frac{1}{(x-4b)^2}\right]\hat{\imath}} \quad \underline{x > 4b}$$

For $b < x < 4b$, $\vec{E}$ from the charge at $x = 4b$ changes direction

$$b<x<4b \quad \vec{E} = \frac{Q}{4\pi\varepsilon_0 x^2}\hat{\imath} - \frac{Q}{4\pi\varepsilon_0 (4b-x)^2}\hat{\imath} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{x^2} - \frac{1}{(4b-x)^2}\right]\hat{\imath}$$

$$\boxed{\vec{E}(x) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{x^2} - \frac{1}{(4b-x)^2}\right]\hat{\imath}} \quad b < x < 4b$$

for $x > 4b$, $\vec{E}$ is in the $+x$ direction and only goes to ϕ at $x \to \infty$. But for $b < x < 4b$, the two contributions to the field tend to cancel. There will be a point at which $\vec{E} = 0$:

$$\frac{1}{x^2} - \frac{1}{(4b-x)^2} = 0$$

$$x^2 = (4b-x)^2$$

$$x = 4b - x$$

$$2x = 4b$$

$$\boxed{x = 2b}$$ $\vec{E} = 0$ at $x = 2b$, midway between the charges.

(e) The total potential $V(x)$ is the scalar sum of the two contributions

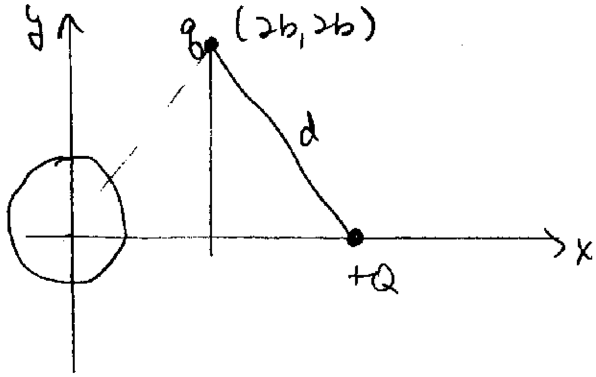
$x > 4b$: $$V(x) = \frac{Q}{4\pi\varepsilon_0 x} + \frac{Q}{4\pi\varepsilon_0 (x-4b)}$$

$b < x < 4b$ $$V(x) = \frac{Q}{4\pi\varepsilon_0 x} + \frac{Q}{4\pi\varepsilon_0 (4b-x)}$$

These can be combined:

$$\boxed{V(x) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{|x|} + \frac{1}{|4b-x|}\right]}$$ $x > b$

(f)



First we need to determine the potential at $x = 2b, y = 2b$

$$V(2b, 2b) = \frac{2Q}{4\pi\varepsilon_0 d}$$ with $d = \sqrt{(2b)^2 + (2b)^2} = 2\sqrt{2}\,b$

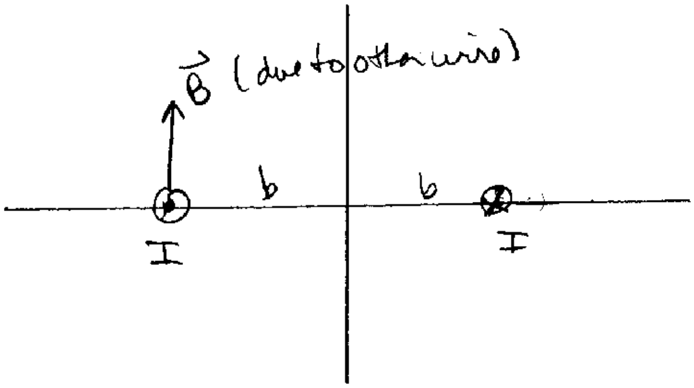
The charge q will then have potential energy

$$PE = \frac{2Qq}{4\pi\varepsilon_0\, 2\sqrt{2}\,b} = \frac{Qq}{4\pi\varepsilon_0 \sqrt{2}\,b}.$$

When the charge is released, all PE will be converted to KE when q reaches ∞

$$\boxed{KE \text{ at } \infty = \frac{Qq}{4\pi\varepsilon_0\sqrt{2}\,b} = \frac{kQq}{\sqrt{2}\,b}}$$

II.



(a) The $\vec{B}$ field due to the wire at $x = +b$ is given by Ampere's Law. $\vec{B}$ forms concentric loops about the wire in the ccw sense.

$|B| = \frac{\mu_0 I}{2\pi r}$, where r is the distance from the wire.

At $x = -b$, the location of the other wire, $\vec{B}$ is vertically upward and $|B| = \frac{\mu_0 I}{2\pi(2b)}$.

The force on the wire at $x = -b$ is (length L)

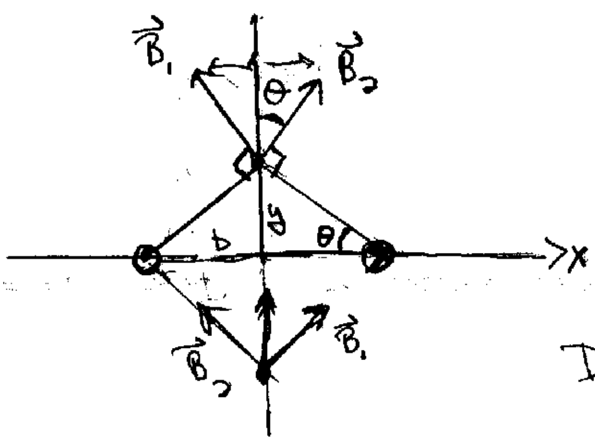
$$\vec{F} = I\vec{L} \times \vec{B} = IL\left(\frac{\mu_0 I}{4\pi b}\right)\underbrace{(-\hat{\imath})}_{}$$

direction is to the left

$$\boxed{\frac{\vec{F}}{L} = \frac{-\mu_0 I^2}{4\pi b}\,\hat{\imath}}$$

force is repulsive.

(b)



The two contributions are $\vec{B}_1$ & $\vec{B}_2$ as shown.

x – components cancel

y – components add

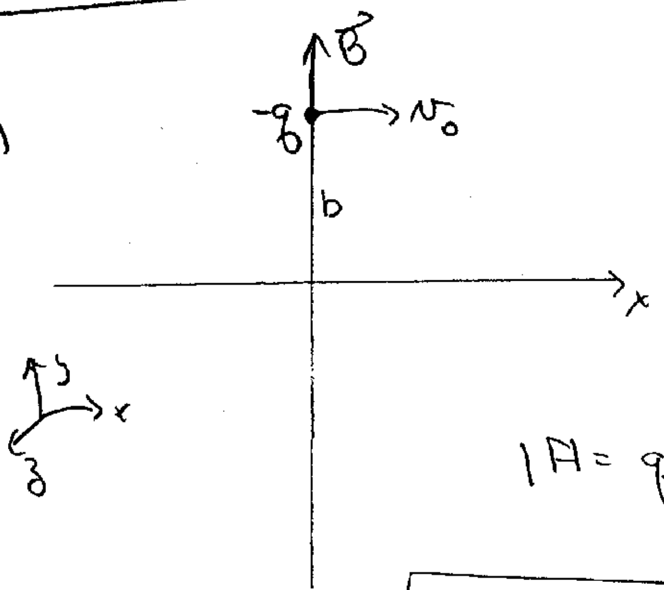
Direction is the same for $\pm y$, positive y-direction.

$$\vec{B}_{TOT} = \vec{B}_1 + \vec{B}_2 = \frac{\cancel{2}\mu_0 I \cos\theta}{\cancel{2}\pi d}\,\hat{j}$$ with $d = \sqrt{b^2+y^2}$ and $\cos\theta = \frac{b}{d}$

$$\boxed{\vec{B}_{TOT} = \frac{\mu_0 I b}{\pi(b^2+y^2)}\,\hat{j}}$$

This expression is valid for both $\pm y$.

(c)



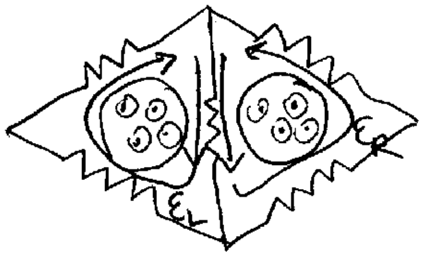
$\vec{F}$ on $q = -q\vec{v}\times\vec{B}$

Direction will be <u>into</u> the page.

$$|F| = q v_0 B = q v_0 \left(\frac{\mu_0 I b}{\pi(b^2+y^2)}\right) = ma$$

$$\boxed{\vec{a} = \frac{q v_0 \mu_0 I b}{\pi m (b^2+y^2)}(-\hat{k})}$$

III.



(a) $\mathcal{E}_L = -\dfrac{d\phi_{BL}}{dt}$

$\phi_{BL} = \pi r^2 B_L$ with $r = 5$ cm

$\dfrac{d\phi_{BL}}{dt} = \pi r^2 \dfrac{dB_L}{dt}$ with $\dfrac{dB_L}{dt} = +60\,\text{mT/s}$

$\mathcal{E}_L = -\pi r^2 \dfrac{dB_L}{dt}$

Since $|B|$ is increasing in the left-hand solenoid, the direction of $\mathcal{E}_L$ will be to produce a downward field $\Rightarrow$ $\mathcal{E}_L$ is clockwise as shown

$\mathcal{E}_L = \pi(.05\,m)^2\ (60\times10^{-3}\,T/s) = 0.47\times10^{-3}\ \dfrac{Ns}{Cm}\cdot\dfrac{m^2}{s} = J/C$

$$\boxed{\mathcal{E}_L = 4.71\times10^{-4}\ V \quad \text{Clockwise}}$$

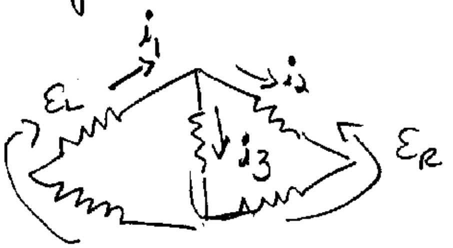
(b) $\dfrac{dB_R}{dt} = -25\,\text{mT/s}$ ($|B|_R$ is decreasing)

$\mathcal{E}_R$ will therefore be in a direction to produce an upward field, counterclockwise.

$\mathcal{E}_R = -\dfrac{d\phi_{BR}}{dt} = \pi r^2 \dfrac{dB_R}{dt} = \pi(.05)^2(25\times10^{-3}\,T/s) = 1.96\times10^{-4}\,V$

$$\boxed{\mathcal{E}_R = 1.96\times10^{-4}\,V \quad \text{CCW}}$$

(c) The current in the central wire can be determined from Kirchhoff's laws: $(R = 20\,\Omega)$



$$i_1 = i_2 + i_3$$

$$\mathcal{E}_L = i_1 R + i_3 R + i_1 R \qquad \mathcal{E}_R = i_3 R - 2 i_2 R$$

$$\mathcal{E}_L = 2 i_1 R + i_3 R \qquad = i_3 R - 2 i_1 R + 2 i_3 R$$

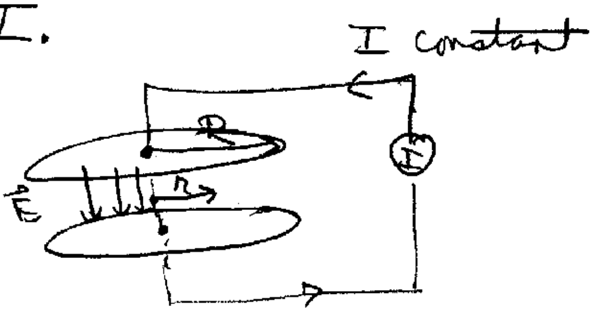
$$\oplus\ \mathcal{E}_R = -2 i_1 R + 3 i_3 R \qquad \mathcal{E}_R = 3 i_3 R - 2 i_1 R$$

$$\mathcal{E}_L + \mathcal{E}_R = 4 i_3 R$$

$$i_3 = \frac{\mathcal{E}_L + \mathcal{E}_R}{4R} = \frac{(4.71 \times 10^{-4}\,V) + (1.96 \times 10^{-4}\,V)}{80\,\Omega} = .083 \times 10^{-4}\,A$$

$$\boxed{i_3 = 8.3\,\mu A}$$ downward in this picture

IV.



(a) $|E| = \frac{\sigma}{\varepsilon_0}$, direction is vertically downward.

(b) $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} = \frac{Q}{\pi R^2 \varepsilon_0}$

$$\frac{dE}{dt} = \frac{dQ/dt}{A\varepsilon_0} = \frac{I}{\pi R^2 \varepsilon_0}$$

$$\boxed{\frac{dE}{dt} = \frac{I}{\pi R^2 \varepsilon_0}}$$

(c) The displacement current is (extension of Ampere's Law)

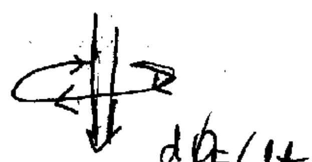
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

For $r < R$, $\Phi_E = \pi r^2 E$

$$\frac{d\Phi_E}{dt} = \pi r^2 \frac{dE}{dt} = \frac{\cancel{\pi} r^2 I}{\cancel{\pi} R^2 \varepsilon_0}$$

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r B$$

$\vec{B}$ forms concentric loops about $\frac{d\Phi_E}{dt}$



Direction is cw as viewed from the top (+) plate.

$$2\pi r B = \frac{\mu_0 \epsilon_0 I r^2}{\epsilon_0 R^2}$$

$$\boxed{|\vec{B}| = \frac{\mu_0 I r}{2\pi R^2}}$$ direction $\underline{\text{cw loops}}$ as viewed from above $\underline{\underline{r < R}}$

(d) for $r = R$ and $r > R$

$$\Phi_E = \pi R^2 E \qquad \text{(E extends only to R)}$$

$$\frac{d\Phi_E}{dt} = \pi R^2 \frac{dE}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 \epsilon_0 \left(\frac{\pi R^2 I}{\pi R^2 \epsilon_0} \right) = \mu_0 I$$

$$\boxed{|\vec{B}| = \frac{\mu_0 I}{2\pi r}}$$ Direction again $\underline{\text{cw}}$ as viewed from the top

$\underline{r \geq R}$

(e)