Physics 102– Pledged Problem 5 Solution

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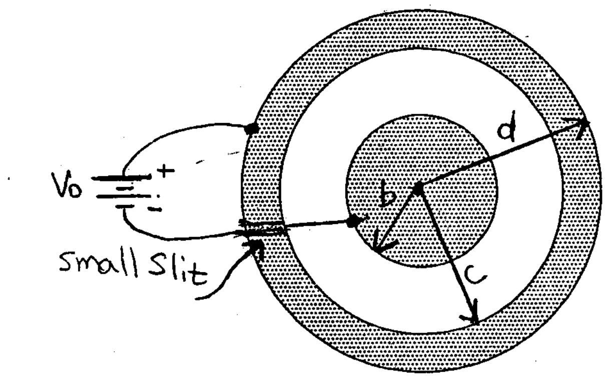
20 I. A solid spherical conductor has radius b. A conducting spherical shell, concentric with the solid sphere, has inner radius c and outer radius d. These two conductors are connected across a battery of potential V_o, with the inner conductor connected to the negative terminal and the outer conductor connected to the positive terminal. Express your answers in terms of V_0, b, c, d, and possibly other constants.

5 (a) Determine the charge on the surface of the inner conductor ($r = b$), and on the inner surface of the outer conductor ($r = c$).

5 (b) Determine the electric field between the conducting shells. Be sure to indicate the direction as well as the magnitude.

2 (c) Determine the capacitance of this configuration and the energy stored in the capacitor.

8 (d) Determine the total energy stored in the electric field between the conducting shells by integrating the energy density of the electric field over that region. Compare to your result from (c).



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(a) To determine the charge on the surfaces of the conductors, recall that

$$\Delta V = -\int \vec{E} \cdot \vec{dl}$$

We know that $\Delta V = V_o$, the voltage of the battery. We also know from the spherical symmetry and Gauss' Law that the electric field will have the form $\vec{E} = \frac{Q}{4\pi\epsilon_o r^2}\hat{r}$. We also know that, since the outer conductor is at a higher potential than the inner conductor, that the electric field will point *inward* toward the center conductor. If we move from $r = b$ to $r = c$, we are moving *against* the field and the potential will be increasing.

Now we can integrate the electric field from to get ΔV, set that equal to V_o, and solve for the charge.

$$V_o = -\int \vec{E} \cdot \vec{dl} = +\frac{Q}{4\pi\epsilon_o} \int_b^c \frac{dr}{r^2}$$

$$V_o = -\frac{Q}{4\pi\epsilon_o}\left[\frac{1}{r}\right]_b^c + \frac{Q}{4\pi\epsilon_o}\left[\frac{1}{b} - \frac{1}{c}\right]$$

$$V_o = +\frac{Q}{4\pi\epsilon_o}\left[\frac{c-b}{bc}\right]$$

Note that ΔV is positive, as required. Now we can solve for Q:

$$\boxed{Q = 4\pi\epsilon_o V_o\left[\frac{bc}{c-b}\right]}$$

This is the *magnitude* of the charge. The surface at $r = b$ must have a negative charge $-Q$, and the surface at $r = c$ must have a positive charge $+Q$. As a cross-check, note that a Gaussian surface drawn inside the outer conductor ($c < r < d$) will enclose zero charge, as required for a conductor.

(b) We already know the form $\vec{E}$ must take, and from (a) we know the charge Q, so we can just write down the field:

$$\boxed{\vec{E} = -\frac{Q}{4\pi\epsilon_o r^2}\hat{r} = -\frac{V_o bc}{(c-b)r^2}\hat{r}}$$

The minus sign means that the electric field is pointing radially inward.

(c) From the definition of capacitance, $C = Q/V$, we can determine C by just dividing out V_o from Q,

$$\boxed{C = Q/V = \frac{4\pi\epsilon_o bc}{c-b}}$$

The total energy stored in a capacitor is $U = \frac{1}{2}CV^2$, so

$$\boxed{U = \frac{2\pi\epsilon_o bcV_o^2}{c-b}}$$

(d) Now determine the energy density in the field and integrate over the region between the two conductors. Recall that the energy density is given by $u_E = \frac{1}{2}\epsilon_o E^2$

$$u_E = \frac{\epsilon_o b^2c^2V_o^2}{2(c-b)^2r^4}$$

$$U = 4\pi \int u_E r^2 dr = \frac{2\pi\epsilon_o b^2c^2V_o^2}{(c-b)^2}\int_b^c \frac{dr}{r^2}$$

$$U = \frac{2\pi\epsilon_o b^2c^2V_o^2}{(c-b)^2}\left[\frac{1}{b} - \frac{1}{c}\right] = \frac{2\pi\epsilon_o b^2c^2V_o^2}{(c-b)^2}\left[\frac{c-b}{bc}\right]$$

$$\boxed{U = \frac{2\pi\epsilon_o bcV_o^2}{(c-b)}}$$

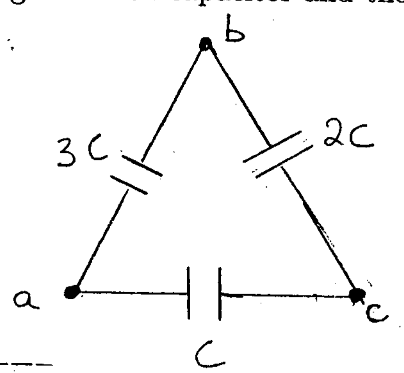
which is the same result as in (c)! We can think of the energy stored in a capacitor as being stored in the electric field.

10 II. Three capacitors are connected to form a triangle as shown below.

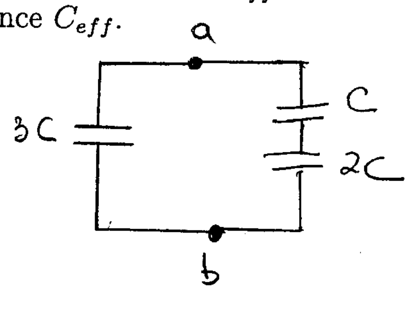
3 (a) Determine the capacitance between points a and b.

3 (b) Determine the capacitance between the points a and c.

4 (c) If a battery V_o is connected between points a and b, determine the charge on each capacitor and the energy stored in each capacitor.



(a) The key idea in understanding this problem is to see which capacitors are in parallel and which are in series. When we look between the points a and b, we can ignore point c, and then we see that the capacitors C and $2C$ are in series. Combine them in the usual way for series capacitors to get the effective capacitance C_{eff1} Then realize that these two are in parallel with $3C$, and combine C_{eff1} with $3C$ in the usual way for capacitors in parallel to get the total effective capacitance C_{eff}.



$$\frac{1}{C_{eff1}} = \frac{1}{C} + \frac{1}{2C}$$

$$C_{eff1} = 2C/3$$

Now combine this with $3C$ to get the final effective capacitance C_{eff},

$$\boxed{C_{eff} = 2C/3 + 3C = 11C/3}$$

(b) To determine the effective capacitance between points a and c, use the same technique. Now $2C$ and $3C$ are in series, giving an effective capacitance of $6C/5$. Then add this in parallel to C, resulting in a final effective capacitance

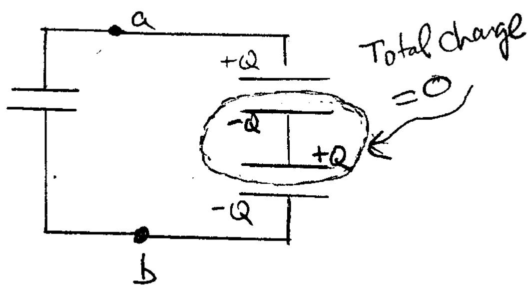
$$\boxed{C_{eff} = 6C/5 + C = 11C/5.}$$

(c) The capacitor $3C$ is directly between points a and b, so it will see the full voltage V_o.

$\boxed{Q_{3C} = 3CV_o}$ and $U_{3C} = \frac{1}{2}(3C)V_o^2 = \boxed{\frac{3CV_o^2}{2}.}$

The other two capacitors (C and $2C$) are in series with each other across the points a and b, and therefore they split the voltage V_o. But they must have the same charge, since charge is conserved on the inner plates (see sketch below). We have already determined the effective capacitance of this series combination, so we can use that result to get the charge:

$C_{eff1} = 2C/3$ so $Q_c = Q_{2c} = C_{eff1}V_o$



$$\boxed{Q_C = Q_{2C} = \frac{2CV_o}{3}}$$

We can get the energy on each of these capacitors by realizing that

$\frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$ which gives

$U_C = \frac{1}{2}(\frac{4C^2V_o^2}{9C}) = \boxed{\frac{2CV_o^2}{9}}$

Likewise for the capacitor $2C$,

$U_{2C} = \frac{1}{2}(\frac{4C^2V_o^2}{9})(\frac{1}{2C}) = \boxed{\frac{CV_o^2}{9}}$

You weren't asked to find this, but it interesting to note that the total energy stored in all three capacitors is

$$U_{tot} = CV_o^2[\frac{2}{9} + \frac{1}{9} + \frac{3}{2}] = \frac{11CV_o^2}{6} = \frac{1}{2}C_{eff}V_o^2$$

where C_{eff} is the total effective capacitance you found in (a).