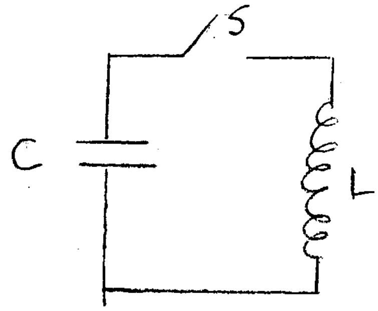
Physics 102– Pledged Problem 10

Time allowed: **2 hours at a single sitting**
Due 5PM Wednesday, April 25, 2007, in the boxes marked Phys 101-102 in the physics lounge. You may use your own textbook, your notes, and a non-programmed calculator. You may also consult the on-line solutions to the corresponding suggested problems. You should consult no other help. Show how you arrived at your answer; the correct answer by itself may not be sufficient.

Further instructions:

(a) Write legibly on **one** side of 8.5" x 11" white or lightly tinted paper.
(b) Staple all sheets together, including this one, in the upper left corner. Make one vertical fold.
(c) On the outside, print your name in capital letters, your LAST NAME followed by your FIRST NAME.
(d) Below your name, print the phrase "Pledged Problem 10", followed by the due date.
(e) Write and sign the pledge, with the understanding that you may consult the materials noted above.
(f) Indicate your **start time** and **end time.**

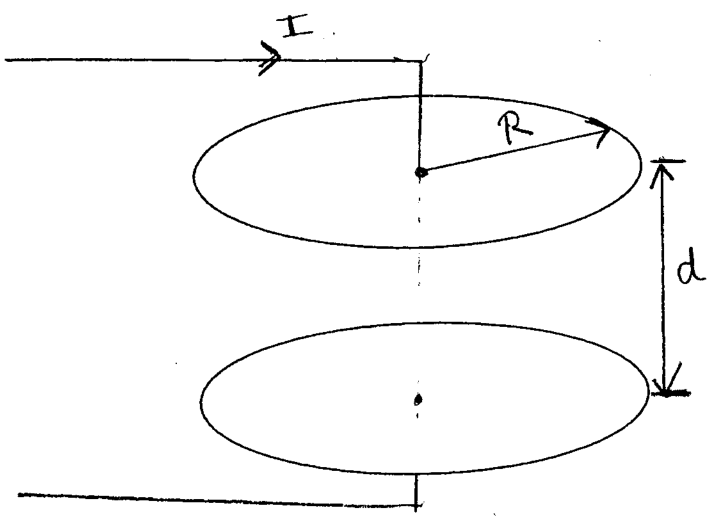
I. A capacitor C and an inductor L are connected in parallel as shown to form an LC oscillator. Initially the capacitor is charged to a voltage V_0 and the switch S is opened. At $t = 0$ the switch is closed. Neglect any resistance in the wires or in the inductor.



(a) Determine the energy stored in the capacitor at $t = 0$.
(b) Determine the period of oscillation T of the circuit.
(c) At $t = T/4$, 1/4 of the way through the cycle, determine
  (i) The energy stored in the capacitor
  (ii) The energy stored in the inductor
  (iii) The current through the inductor
  (iv) The total energy stored in the circuit.
(d) Repeat (c) for $t = T/2$, half way through the oscillation cycle.
(e) Describe qualitatively what happens if a small resistance R is included in the circuit.

II. A very large cylindrical capacitor of radius R and plate separation d is being charged slowly with constant current I. As the capacitor charges, the electric field between the plates increases with time. Take r to be the radial distance from the axis of the capacitor, as shown in the figure below.

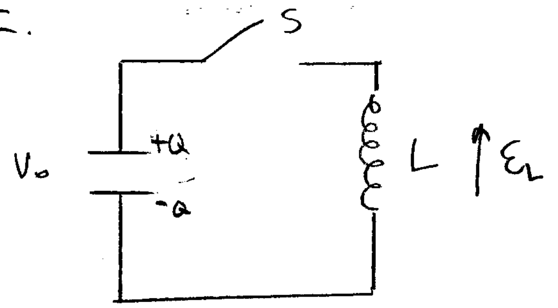
(a) At a particular point in time, the surface charge density on the plates is σ. What is the electric field $\vec{E}$ between the plates at that time?
(b) Determine the time rate of change of the electric field, $\frac{dE}{dt}$, in terms of I, R, and other constants.
(c) Near the center of the plates, the electric field is constant in space. Determine the magnetic field $\vec{B}(r)$ between the plates for $r < R$ in terms of I, R and other constants. Indicate both direction and magnitude of the field.
(d) Neglecting fringe effects around the edge of the capacitor plates, determine the magnetic field $\vec{B}(r)$ for $r = R$ and for $r > R$. Sketch $B(r)$ for all r.



Phys 102

Pledged Problem 10

I.



(a) $\boxed{U_c = \frac{1}{2} C V_0^2}$

(b) If the charge on the capacitor is Q, the voltage drop across C is $\frac{Q}{C}$. When the switch is closed, I is in the clockwise direction as is $\frac{dI}{dt}$. But $I = -\frac{dQ}{dt}$, since the charge on C is decreasing. Then Kirchoff's loop equation is

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0 \quad \Rightarrow \quad \frac{d^2Q}{dt^2} + \underbrace{\frac{1}{LC}}_{\omega^2} Q = 0$$

This last eqn. is exactly the form for simple harmonic oscillations with $\omega^2 = \frac{1}{LC}$

$$\omega = 2\pi\nu = \frac{2\pi}{T} = \frac{1}{\sqrt{LC}} \quad \text{or} \quad \boxed{T = 2\pi\sqrt{LC}}$$

(c) The equation for Q is then

$$Q(t) = A\cos\omega t + B\sin\omega t$$

$$Q(t=0) = CV_0 = A$$

$$\frac{dQ(t)}{dt} = -I(t) = 0 = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$0 = \omega B \quad \text{or } B = 0$$

$$\boxed{Q(t) = CV_0\cos\omega t.}$$

At $t = T/4$, $\omega t = \frac{1}{\sqrt{LC}}\left(\frac{2\pi\sqrt{LC}}{4}\right)$

$$\omega t = \frac{2\pi}{4} = \pi/2$$ — This makes sense since one full cycle has $\omega t = 2\pi$

$Q(T/4) = CV_0\cos(\pi/2) = 0$ => no charge on the capacitor,

so $\boxed{U_c = 0 \text{ at } t = T/4}$ (i)

$$\frac{dQ}{dt} = I = -CV_0\omega\sin\omega t$$

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}L\left(C^2V_0^2\omega^2\underbrace{\sin(\pi/2)}_{=1}\right)$$

$$U_L = \frac{1}{2}\cancel{L}\frac{C^2V_0^2}{\cancel{L}C} = \frac{1}{2}CV_0^2$$

$\boxed{U_L = \frac{1}{2}CV_0^2}$ (ii) — at $t = T/4$, all the energy is in the inductor!

$$|I(t = T/4)| = \frac{CV_0}{\sqrt{LC}} \sin(\overbrace{\pi/2}^{=1})$$

$$\boxed{I(T/4) = \frac{CV_0}{\sqrt{LC}}}$$ (iii) Direction is clockwise

Total energy is $\boxed{U_C + U_L = \frac{1}{2} C V_0^2}$ (iv)

Same as the initial energy.

(d) At $t = T/2 = \frac{2\pi\sqrt{LC}}{2} = \pi\sqrt{LC}$

$$Q(T/2) = CV_0 \cos\left(\frac{\pi \cancel{\sqrt{LC}}}{\cancel{\sqrt{LC}}}\right) = -CV_0$$

$$U_C = \frac{1}{2} C V^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{C^2 V_0^2}{C} = \frac{1}{2} C V_0^2$$

$\boxed{U_C = \frac{1}{2} C V_0^2}$ (i)

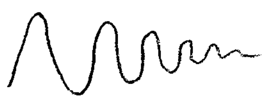
$$I = \frac{dQ}{dt} = -\omega C V_0 \sin\pi = 0.$$

Since $I = 0$, $\boxed{U_L = 0}$ (ii)

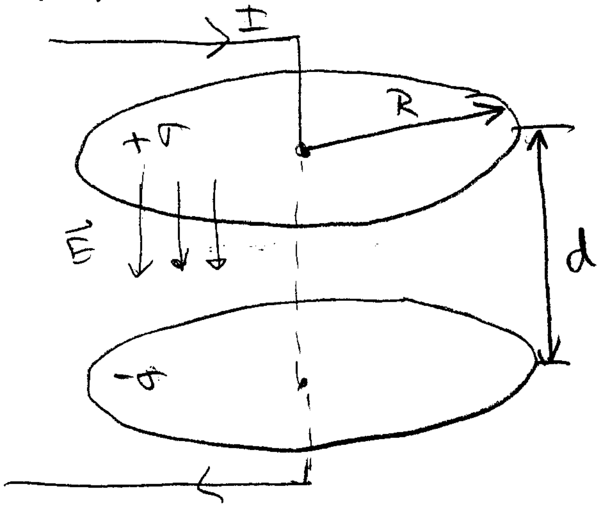
$\boxed{I = 0}$ (iii)

$\boxed{U_{TOT} = U_C + U_L = \frac{1}{2} C V_0^2}$ (iv) — Same as the initial energy!

(e) If there is some resistance R in the circuit, the total energy decays with time due to $I^2 R$ losses in R. The amplitude of the oscillation decreases with time.



II.



(a) When the surface charge on the plates $= \sigma$, by Gauss' law

$$\int \vec{E} \cdot d\vec{A} = EA = \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{|\vec{E}| = \frac{\sigma}{\varepsilon_0}}$$

Direction is $\perp$ to the plates, away from the top plate $(-\hat{j})$.

(b) $I = \dfrac{dQ}{dt}$

Q_{TOT} on the plates $= \sigma(\pi R^2)$

$$\sigma = \frac{Q}{\pi R^2} \qquad \frac{d\sigma}{dt} = \frac{dQ}{dt} \cdot \frac{1}{\pi R^2} = \frac{I}{\pi R^2}$$

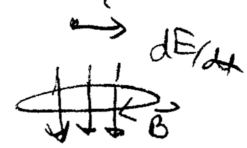
$$\frac{dE}{dt} = \frac{d\sigma}{dt} \cdot \frac{1}{\varepsilon_0} = \frac{I}{\pi R^2 \varepsilon_0}$$

$$\boxed{\left|\frac{d\vec{E}}{dt}\right| = \frac{I}{\pi R^2 \varepsilon_0}}$$

As σ increases, $\vec{E}$ increases downward, in the $-\hat{j}$ direction

(c) From the generalized form of Ampere's law,

$$\oint \vec{B} \cdot d\vec{\ell} = \cancel{\mu_0 I}^{\,0} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$



$$2\cancel{\pi}\cancel{r} B = \mu_0 \varepsilon_0 \frac{d}{dt}\left(E(\cancel{\pi} r^{\cancel{2}})\right) \qquad (\Phi_E = \pi r^2 E)$$

$$|\vec{B}| = \frac{\mu_0 \varepsilon_0}{2} \frac{dE}{dt} = \frac{\mu_0 \cancel{\varepsilon_0}}{2}\left(\frac{I r}{\pi R^2 \cancel{\varepsilon_0}}\right)$$ – B increases linearly w/ r

$$\boxed{|\vec{B}| = \frac{\mu_0 I r}{2\pi R^2}}$$

direction is clockwise loops, as seen from above.

E increasing downward.

(d) For $r = R$

$$\boxed{|\vec{B}| = \frac{\mu_0 I}{2\pi R}}$$ – clockwise loops as seen from above

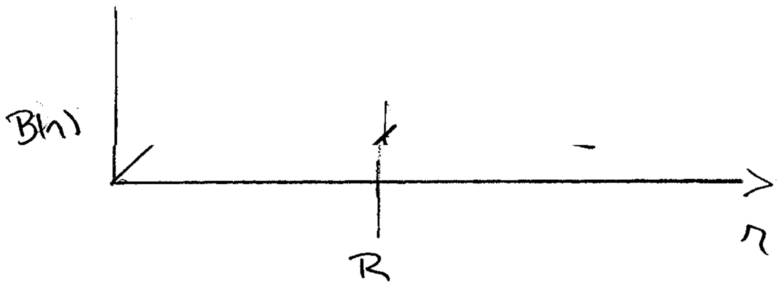
For $r > R$, $\Phi_E = \pi R^2 E$, since we are taking $\vec{E}$ to drop to zero for $r > R$

$$\oint \vec{B} \cdot d\vec{s} = 2\pi r B = \mu_0 \varepsilon_0 \frac{d}{dt}(E \pi R^2)$$

$$= \mu_0 \cancel{\varepsilon_0} (\cancel{\pi R^2}) \left(\frac{I}{\cancel{\pi R^2} \cancel{\varepsilon_0}}\right) = \mu_0 I$$

$$\boxed{|\vec{B}| = \frac{\mu_0 I}{2\pi r}}$$

direction is clockwise loops as seen from above.


B(r)
R
r