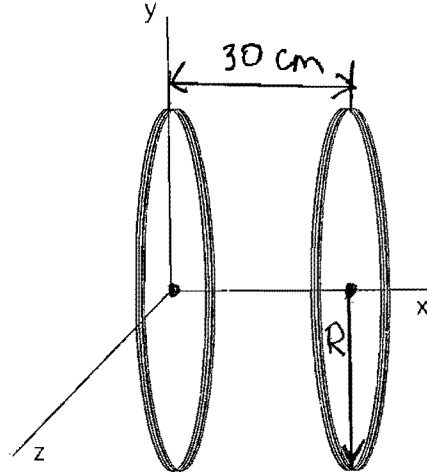


## Physics 102– Pledged Problem 8 Solutions

I. Two circular coils that are separated by a distance equal to their radii and that carry equal currents such that their axial magnetic fields add are called Helmholtz coils. A useful feature of Helmholtz coils is that the resulting magnetic field between the coils is very uniform. Take the radius of the coils to be  $R=30\text{cm}$ , the current through each turn to be  $I=10\text{A}$ , and the number of turns in each coil to be  $N=500$ . Place one coil in the  $yz$  plane with its center at the origin and the other in a parallel plane at  $x=30\text{cm}$ .

- (a) Calculate the resultant field  $B_x$  at  $x=10\text{cm}$ ,  $x=15\text{cm}$ ,  $x=30\text{cm}$ ,  $x=35\text{cm}$ , and  $x=40\text{cm}$ .  
 (b) Use your results and the fact that  $B_x$  is symmetric about the midpoint to sketch  $B_x$  vs.  $x$  from  $x=0$  to  $x=40\text{cm}$ .



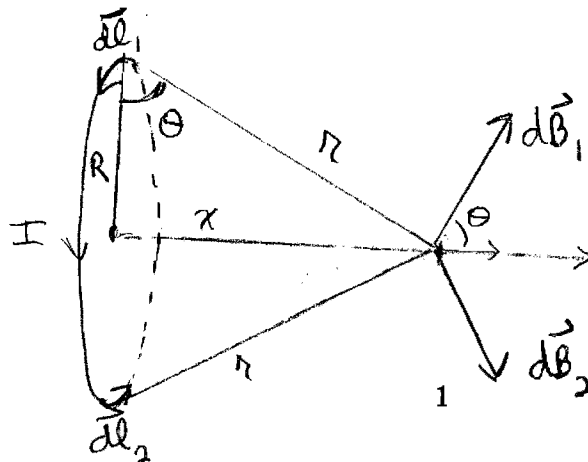
First we need to find the magnetic field due to a circular loop of current. This has been worked out in your book on page 836. The strategy is to use the Biot-Savart law to determine the current due to two small segments of current  $dl_1$  and  $dl_2$ , chosen in such a way as to cancel the components perpendicular to the axis of the loop.

Biot-Savart Law:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$

where  $r$  is the distance from the current element to the location at which the magnetic field is to be determined. In the figure we see that we have  $r = \sqrt{x^2 + R^2}$ . Also in the figure, you can see that the  $y$ -components cancel and the  $x$ -components add. For any current element  $dl$  around the loop, there will be a matching current element on the opposite side that cancels out the  $y$ -components in the same way.

The resulting field has a component only along the axis of the circle, which is the  $x$ -axis in this case. The  $x$ -component is given by  $B \cos \theta$  with  $\cos \theta = \frac{R}{\sqrt{x^2 + R^2}}$ . In the figure  $dl_1$  is coming out of the page, perpendicular to  $\hat{r}$ . So  $\sin \theta$  from the cross product is just 1.



Now apply the Biot-Savart Law:

$$dB_x = dB \cos \theta = \frac{\mu_o}{4\pi} \cdot \frac{R}{\sqrt{x^2+R^2}} \cdot \frac{Idl}{x^2+R^2}$$

Now we integrate over  $dl$ , but  $x$ ,  $R$ , and  $\cos \theta$  are all constant over the integral! So the integral over  $dl$  just becomes the circumference of the circle,  $2\pi R$ . Then we get for  $B_x$

$$B_x = \frac{\mu_o IR^2}{2(x^2+R^2)^{3/2}}$$

This is for just one loop. The Helmholtz coils in this case have 500 turns, so this result is multiplied by 500. The two coils are 0.3m apart, with  $R=0.3m$ . (You must convert to mks units!). For a point a distance  $x$  from the origin, the distance to the center of the right coil is  $0.3m-x$ .

Now we can calculate the field due to the Helmholtz coils at the locations required. We need the numerical value of  $\mu_o = 4\pi \times 10^{-7} N/A^2$ . For the left coil we have:

$$B_{xL}(x = .1m) = \frac{4\pi \times 10^{-7} \frac{N}{A^2} (500)(10A)(0.3m)^2}{2(.01+.09)^{3/2}} = 8.93 \times 10^{-3} \frac{N}{A \cdot m} = 8.93 \times 10^{-3} T.$$

Note that Tesla= $N/(A \cdot m)$ , so the units are correct.

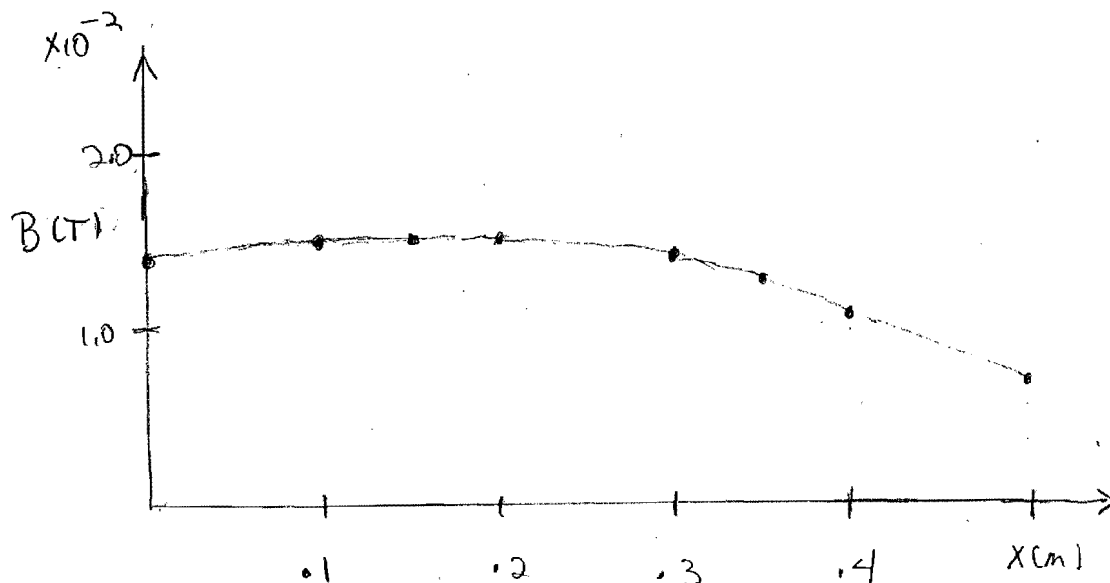
For the right coil the correct value to use in the denominator is 0.2m:

$$B_{xR}(x = .1m) = \frac{4\pi \times 10^{-7} \frac{N}{A^2} (500)(10A)(0.3m)^2}{2(.04+.09)^{3/2}} = 6.03 \times 10^{-3} T$$

Adding the two results give the total field at that point. Then repeat for the other values of  $x$  requested. The values for the magnetic field are:

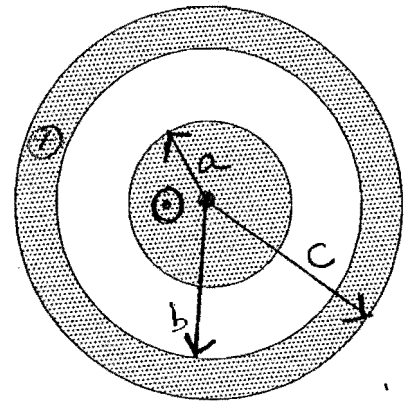
$B(x=0.1m) = 1.496 \times 10^{-2} T$
$B(x=0.15m) = 1.497 \times 10^{-2} T$
$B(x=0.3m) = 1.416 \times 10^{-2} T$
$B(x=0.35m) = 1.292 \times 10^{-2} T$
$B(x=0.4m) = 1.119 \times 10^{-2} T$

(b) See the sketch below. Note that between the coils the field is very uniform, but it falls off quickly past either end.



II. The figure below shows the cross section of a long conductor of a type called a coaxial cable. The radius of the inner solid cylinder is  $a$ , and the outer cylindrical shell has inner radius  $b$  and outer radius  $c$ , as shown in the figure below. The conductors carry equal but opposite currents, with the current in the inner conductor flowing out of the page. The currents are uniformly distributed over the cross-sectional area in each case. The coordinate  $r$  measures the distance from the axis of the cylinders. Determine the magnetic field  $\vec{B}(r)$  in the ranges indicated below, being sure to indicate the direction of  $\vec{B}$  as well as the magnitude.

- (a)  $r < a$
- (b)  $a < r < b$
- (c)  $b < r < c$
- (d)  $r > c$ .
- (e) Sketch the magnitude of the magnetic field  $B(r)$  as a function of  $r$ .



Due to the symmetry, we can easily solve this problem using Ampere's Law.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

where  $I_{encl}$  is the current that passes through the two-dimensional area enclosed by the contour.

From the Biot-Savart Law we know the magnetic field forms concentric loops around the wire, and due to the symmetry we know that the field at all points on the loop must have the same magnitude. We choose the contour to be concentric loops around the wire, and then the integral is

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B = \mu_0 I_{encl}$$

$$B = \frac{\mu_0 I_{encl}}{2\pi r} \quad \text{where } r \text{ is the distance from the center of the wire}$$

(a) For  $r < a$ , not all of the current is enclosed by the contour. We need to determine the *current density*  $J$ , which is the current per unit area,  $J = I/A$ . Then the current enclosed is  $J$  times the area enclosed. For  $r < a$ ,  $J = \frac{I}{\pi a^2}$ , and

$$2\pi r B = \mu_0 J \pi r^2 = \mu_0 \frac{I \pi r^2}{\pi a^2}$$

$$B = \frac{\mu_0 I r}{2\pi a^2} \quad \text{where the direction is counter-clockwise loops, as given by the right-hand rule.}$$

(b) For  $a < r < b$ , all of the inner current is enclosed and none of the outer current, so the field in this region is just

$$B = \frac{\mu_0 I}{2\pi r} \quad \text{with the direction counter-clockwise loops.$$

(c) For  $b < r < c$ , part of the outer current (which is in the opposite direction) is enclosed, so we have to determine the current density  $J$  in this case. The full current is still  $I$ , but now the area is  $\pi c^2 - \pi b^2$ , giving a current density of

$$J = \frac{I}{\pi(c^2 - b^2)}$$

The magnetic field is now given by

$$2\pi r B = \mu_0 I - \mu_0 J \pi (r^2 - b^2)$$

$$B = \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I (r^2 - b^2)}{2\pi r (c^2 - b^2)}$$

$$B = \frac{\mu_0 I}{2\pi r} \left[ 1 - \frac{(r^2 - b^2)}{(c^2 - b^2)} \right] \quad \text{with the direction still counterclockwise loops.$$

(d) For  $r > c$ , all of the outer current (which is in the opposite direction as the inner current) is enclosed, so the net current is zero and  $B = 0$ .

(e) See sketch below. Note that for  $r < a$  the field is increasing linearly. For  $a < r < b$ , the field falls as  $1/r$ . In the region  $b < r < c$ , the field falls off more quickly than  $1/r$ .

