Physics 102– Pledged Problem 5

Time allowed: **2 hours at a single sitting**

Due 5PM Monday, February 27, 2006, in the boxes marked Phys 101-102 in the physics lounge. You may use your own textbook, your notes, and a non-programmed calculator. You may also consult the on-line solutions to the corresponding suggested problems. You should consult no other help. Show how you arrived at your answer; the correct answer by itself may not be sufficient.

Further instructions:

(a) Write legibly on **one** side of 8.5" x 11" white or lightly tinted paper.

(b) Staple all sheets together, including this one, in the upper left corner and make one vertical fold.

(c) On the outside, staple side up, print your name in capital letters, your LAST NAME first followed by your FIRST NAME.

(d) Below your name, print the phrase "Pledged Problem 5", followed by the due date.

(e) Also indicate **start time** and **end time**.

(f) Write and sign the pledge, with the understanding that you may consult the materials noted above.

I. An insulating sphere of radius R has a total positive charge Q uniformly distributed throughout its volume. The coordinate r measures the distance from the center of the sphere.

(a) Determine the energy density $u_E(r)$ of the electric field for $r < R$.

(b) Integrate $u_E(r)$ over the volume of the sphere to determine the total energy stored in the electric field inside the sphere.

(c) Determine the energy density $u_E(r)$ of the electric field for $r > R$.

(d) Integrate $u_E(r)$ for r from $R \to \infty$ to determine the total energy stored in the electric field for $r > R$.

(e) Check that the units in your answers in (b) and (d) are correct, that is that they work out to units of Joules.

(f) A proton has radius of $R = 10^{-15}$ meters, a unit known as a Fermi, also called a femtometer. The charge on a proton is 1.6 $\times 10^{-19}$ Coulomb. From your result in (b) and (d), calculate the energy stored in the electric field of the proton, assuming the charge is uniformly spread throughout the volume of the proton. Compare this to the rest mass energy of the proton, where $E = mc^2$. The mass of the proton is 1.67 $\times 10^{-27}$ kg and the speed of light $c = 3 \times 10^8$ m/s.

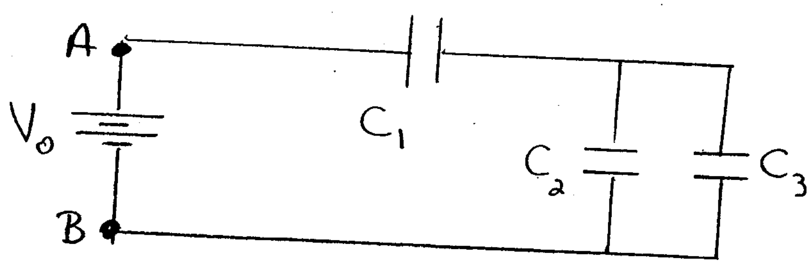
II. Consider the arrangement of capacitors shown below. The three capapcitors have values $C_1 = C_2 = C$ and $C_3 = 2C$. A battery of voltage V_o is connected across the points A and B. The capacitors are allowed to charge up fully.

(a) Determine the effective capacitance C_{eff} of this configuration and the total charge supplied by the battery.

(b) Determine the charge on each capacitor and the voltage across each capacitor.

(c) Determine the energy stored in each capacitor.

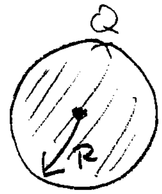
(d) The battery remains connected and a dielectric of dielectric constant κ is inserted into C_1, completely filling the space between the capacitor plates. Redo (a)-(c) for this situation.



Phys 102

Pledged Problem 5

I.



$$\rho = \frac{3Q}{4\pi R^3} = \text{constant}$$

(a) Use Gauss' Law to determine the electric field for $r < R$

$$\oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho(\frac{4}{3}\pi r^3)}{\varepsilon_0}$$

$$\vec{E} = \frac{\rho r}{3\varepsilon_0} \hat{r}$$

$$\boxed{u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{\rho^2 r^2}{18\varepsilon_0}}$$

Or, in terms of Q

$$u_E = \frac{9Q^2 r^2}{16\pi^2 R^6 (18\varepsilon_0)} = \boxed{\frac{Q^2 r^2}{32\pi^2 \varepsilon_0 R^6} = u_E}$$

(b) Integrate u_E over the volume of the sphere to get the total energy stored in the electric field inside the sphere.

$$U_E = \int u_E \, dV = \frac{\rho^2}{18\varepsilon_0} \int_0^R r^2 \underbrace{(4\pi r^2 dr)}_{dV}$$

$$U_E = \frac{4\pi \rho^2}{18\varepsilon_0} \int_0^R r^4 dr$$

$$= \frac{\cancel{4}^{2}\pi \rho^2}{\cancel{18}_{9}\varepsilon_0} \cdot \frac{R^5}{5} = \frac{2\pi\rho^2 R^5}{45\varepsilon_0}$$

$$\boxed{U_E = \frac{2\pi\rho^2 R^5}{45\varepsilon_0}}$$

or in terms of Q:

$$U_E = \frac{\cancel{2\pi} R^{\cancel{5}}}{\cancel{45}_{15}\varepsilon_0} \cdot \frac{\cancel{9}^{3} Q^2}{\cancel{16\pi^2}_{8\pi} R^{\cancel{6}}} = \frac{3Q^2}{8\pi\varepsilon_0(15)R} = \frac{Q^2}{40\pi\varepsilon_0 R}$$

$$\boxed{U_E = \frac{Q^2}{40\pi\varepsilon_0 R}}$$

(c) for $r > R$, from Gauss' Law we know

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$u_E(r>R) = \frac{1}{2}\varepsilon_0 E^2 = \frac{Q^2}{16\pi^2\varepsilon_0^4 r^4} \cdot \frac{\varepsilon_0}{2}$$

$$\boxed{u_E(r>R) = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4}}$$

(d) $U_E(r>R) = \int_R^{\infty} u_E(r>R)\, dV$

$$= \frac{Q^2}{32\pi^2\varepsilon_0} \int_R^{\infty} \frac{1}{r^4} \overbrace{(4\pi r^2\, dr)}^{dV}$$

$$= \frac{4\pi Q^2}{32\pi^2 \varepsilon_0} \int_R^{\infty} \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0} \left(-\frac{1}{r} \right) \Bigg|_R^{\infty}$$

$$\boxed{U_E(r>R) = \frac{Q^2}{8\pi\varepsilon_0 R}}$$

(e) Check units – always a good sanity check!

In (b) we had

$$U_E = \frac{Q^2}{40\pi\varepsilon_0 R}$$

The units of ε_0 can be determined from Coulomb's law.

$$F = \frac{Q^2}{4\pi\varepsilon_0 r^2} \qquad [\varepsilon_0] = \frac{\text{Coul}^2}{\text{N-m}^2}$$

$$[U_E] = \frac{\cancel{\text{Coul}^2}\ \text{N-m}^{\cancel{2}}}{\cancel{\text{Coul}^2}\ \cancel{\text{m}}}$$

$$= \text{N-m} = \text{Joule} \checkmark$$

U_E has units of energy as required.

The answer in (d) has the same units – just a different constant.

We can also check the units of the answer in (b) using ρ (Coul/m³)

$$[U_E] = \frac{\cancel{Coul}^2}{\cancel{m^6}} \cdot \frac{\cancel{m^5}\ N\text{-}m^2}{\cancel{Coul^2}} = N\text{-}m = \text{Joule} \checkmark$$

(f) $R_p = 10^{-15}\,m \qquad Q = 1.6\times10^{-19}\,C$

$$U_{TOT} = U(r<R) + U(r>R)$$

$$U_{TOT} = \frac{Q^2}{40\pi\epsilon_0 R} + \frac{5Q^2}{5\cdot 8\pi\epsilon_0 R} = \frac{6Q^2}{40\pi\epsilon_0 R} \qquad \epsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N\text{-}m^2}$$

$$U_{TOT} = \frac{6\,(1.6\times10^{-19}\,C)^2\ N\text{-}m^2}{40\pi\,(10^{-15}\,m)(8.85\times10^{-12}\,C^2)} = \frac{15.4\times10^{-38}}{1112.1\times10^{-27}}\ \frac{\cancel{C^2}\ N\text{-}m^2}{\cancel{C^2}\text{-}m}$$

$$U_{TOT} = \frac{1.54\times10^{-37}\,J}{1.11\times10^{-24}} = \boxed{1.4\times10^{-13}\,J = U_{TOT}}$$ ← total energy in the electric field

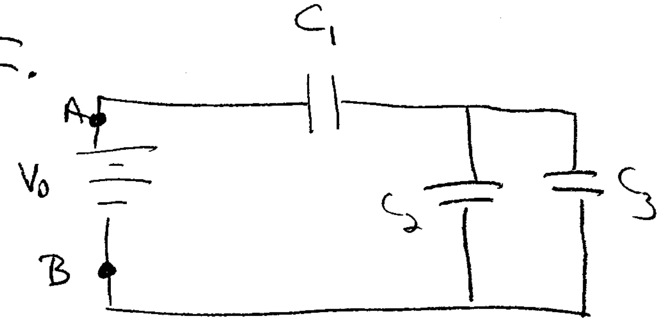
Compare to the rest mass energy of the proton

$$E = mc^2 = (1.67\times10^{-27}\,kg)(3\times10^8\,m/s)^2 = 15.03\times10^{-11}\ \text{kg m}^2/s^2 \;(= J)$$

$$\boxed{E = 1.5\times10^{-10}\,J = \text{rest mass energy of proton}}$$

So the rest mass energy of the proton is much larger than the energy contained in the electric field.

II.



$C_1 = C_2 = C$

$C_3 = 2C$

(a) first combine C_2 & C_3 – add in parallel

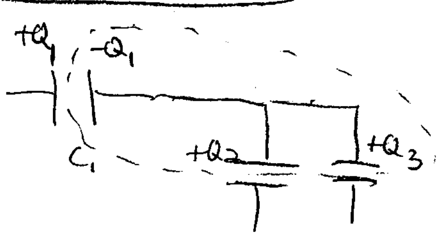
$$C_{eff\,23} = C_2 + C_3 = 3C$$

then combine $C_{eff\,23}$ with C_1 in series.

$$\frac{1}{C_{eff}} = \frac{1}{C_1} + \frac{1}{C_{eff\,23}} = \frac{1}{C} + \frac{1}{3C} = \frac{4}{3C}$$

$$\boxed{C_{eff} = \frac{3C}{4}}$$

(b)



$Q_2 + Q_3 = Q_1$

$V_2 = V_3$ since they are in parallel

$V_1 + V_2 = V_1 + V_3 = V_0$

$$Q_1 = C_{eff} V_0 = \frac{3CV_0}{4}$$

$$V_1 = \frac{Q_1}{C} = \frac{3V_0}{4}$$

$$V_2 = V_3 = \frac{V_0}{4}$$

$$Q_2 = V_2 C_2 = \frac{V_0 C}{4}$$

$$Q_3 = V_3 C_3 = \frac{V_0}{4} \cdot 2C = \frac{V_0 C}{2}$$

$$Q_2 + Q_3 = \frac{3V_0 C}{2} = Q_1 \text{ as required}$$

Summarize:

$$\boxed{\begin{array}{ll} Q_1 = \frac{3CV_0}{4} & V_1 = \frac{3V_0}{4} \\ Q_2 = \frac{V_0 C}{4} & V_2 = \frac{V_0}{4} \\ Q_3 = \frac{V_0 C}{2} & V_3 = \frac{V_0}{4} \end{array}}$$

(c) $U_c = \frac{1}{2} CV^2 = \frac{1}{2} QV$

$U_1 = \frac{1}{2} \cdot \frac{3CV_0}{4} \cdot \frac{3V_0}{4}$

$$\boxed{U_1 = \frac{9V_0^2 C}{32}}$$

$U_2 = \frac{1}{2} \cdot \frac{V_0 C}{4} \cdot \frac{V_0}{4}$

$$\boxed{U_2 = \frac{V_0^2 C}{32}}$$

$U_3 = \frac{1}{2} \cdot \frac{V_0 C}{2} \cdot \frac{V_0}{4}$

$$\boxed{U_3 = \frac{V_0^2 C}{16}}$$

Note: $U_1 + U_2 + U_3 = \frac{3CV_0^2}{8} = \frac{1}{2} C_{eff} V_0^2$

(d) $C_1 \to KC_1 = KC$ the capacitances of C_2 & C_3 are unchanged.

$\frac{1}{C_{eff}} = \frac{1}{KC} + \frac{1}{3C} = \frac{1}{C}\left[\frac{1}{K} + \frac{1}{3}\right] = \frac{1}{C}\left(\frac{3+K}{3K}\right)$

$$\boxed{C_{eff} = \frac{3KC}{3+K}}$$

$V_1 = \frac{Q_1}{KC} = \frac{3\cancel{KC}V_0}{(3+K)(\cancel{KC})} = \frac{3V_0}{3+K}$

$Q_1 = C_{eff} V_0 =$

$$\boxed{Q_1 = \frac{3KCV_0}{3+K}} \qquad \boxed{V_1 = \frac{3V_0}{3+K}} \qquad \boxed{U_1 = \frac{9KCV_0^2}{2(3+K)^2}}$$

$$V_2 = V_3 = V_0 - V_1 = V_0 - \frac{3V_0}{3+K} = \frac{3V_0 + KV_0 - 3V_0}{3+K}$$

$$\boxed{V_2 = V_3 = \frac{KV_0}{3+K}}$$

$$Q_2 = C_2 V_2 = \boxed{\frac{KCV_0}{3+K} = Q_2} \qquad \boxed{u_2 = \frac{CK^2V_0^2}{2(3+K)^2}}$$

$$Q_3 = C_3 V_3 = \boxed{\frac{2KCV_0}{3+K} = Q_3} \qquad \boxed{u_3 = \frac{CK^2V_0^2}{(3+K)^2}}$$

Check:

$$u_{TOT} = \frac{1}{2} C_{eff} V_0^2 = \frac{3KCV_0^2}{2(3+K)}$$

$$u_1 + u_2 + u_3 = \frac{KCV_0^2}{(3+K)^2}\left[\frac{9}{2} + \frac{K}{2} + K\right]$$

$$\frac{KCV_0^2}{2(3+K)^2}\left[9+3K\right] = \frac{3KCV_0^2}{2(3+K)}$$