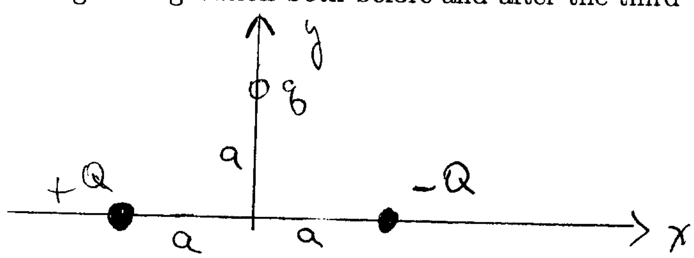
Physics 102– Pledged Problem 4

Time allowed: **2 hours at a single sitting**
Due 5PM Monday, Februaay 12, 2007, in the boxes marked Phys 101-102 in the physics lounge. You may use your own textbook, your notes, and a non-programmed calculator. You may also consult the on-line solutions to the corresponding suggested problems. You should consult no other help. Show how you arrived at your answer; the correct answer by itself may not be sufficient.
Further instructions:
(a) Write legibly on **one** side of 8.5" x 11" white or lightly tinted paper.
(b) Staple all sheets together, including this one, in the upper left corner. Make one vertical fold.
(c) On the outside, print your name in capital letters, your LAST NAME followed by your FIRST NAME.
(d) Below your name, print the phrase "Pledged Problem 4", followed by the due date.
(e) Write and sign the pledge, with the understanding that you may consult the materials noted above.
(f) Indicate your **start time** and **end time**.

I. A positive point charge of $+Q$ is located on the x-axis at $x = -a$. A second negative point charge $-Q$ is located on the x-axis at $x = +a$. Take the zero of the electrostatic potential to be at infinity.

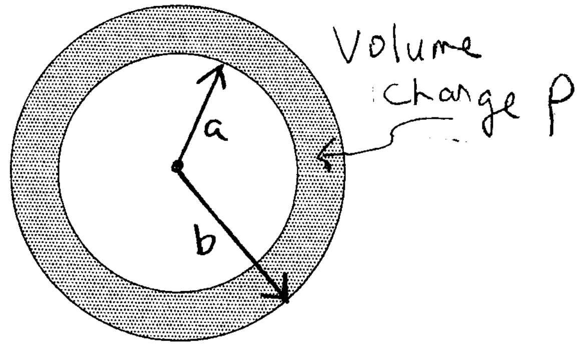
(a) For this charge configuration, determine the electrostatic potential $V(x)$ for all points on the x-axis. Sketch $V(x)$ vs. x.
(b)At what point(s) on the x-axis is $V(x)$=0? What is the electric field $\vec{E}$ at those locations?
(c) Determine the electrostatic potential $V(y)$ for all points along the y-axis.

(d) Now suppose a third charge $+Q$ is moved at constant speed from very far away to the point on the positive y-axis at $y = a$. How much work must be done to move the third charge to this location?
(e) What is the electric field at the location of the third charge, $x = 0, y = a$? How do you reconcile this answer with your answer in (d)?
(f) Determine the total electrostatic potential energy of this charge configuration both before and after the third charge is moved into place.



II. A spherical shell of *nonconducting* material has an inner radius a and outer radius b. It carries a uniform *volume* charge distribution ρ. Take the zero of the electrostatic potential to be zero at infinity.

(a) Determine the total charge Q contained in the spherical shell.
(b) Determine the electrostatic potential $V(r)$ as a function of r for $r > b$.
(c) Determine the electric field $\vec{E}(r)$ for the region $a < r < b$. Using this result for $\vec{E}$, determine the potential $V(r)$ for the region $a < r < b$.
(d) Determine the potential $V(r)$ for the region $r < a$.
(e) Sketch $V(r)$ for all r and indicate any points or regions where the electric field $\vec{E}$ is zero.
(f) If a small positive charge q is released from rest at $r = 2b$, determine its kinetic energy when it is at the location $r = 10b$.



Physics 102– Pledged Problem 4 Solution

I. A positive point charge of $+Q$ is located on the x-axis at $x = -a$. A second negative point charge $-Q$ is located on the x-axis at $x = +a$. Take the zero of the electrostatic potential to be at infinity.

(a) For this charge configuration, determine the electrostatic potential $V(x)$ for all points on the x-axis. Sketch $V(x)$ vs. x.
(b)At what point(s) on the x-axis is $V(x)$=0? What is the electric field $\vec{E}$ at those locations?
(c) Determine the electrostatic potential $V(y)$ for all points along the y-axis.

(d) Now suppose a third charge $+Q$ is moved at constant speed from very far away to the point on the positive y-axis at $y = a$. How much work must be done to move the third charge to this location?
(e) What is the electric field at the location of the third charge, $x = 0, y = a$? How do you reconcile this answer with your answer in (d)?
(f) Determine the total electrostatic potential energy of this charge configuration both before and after the third charge is moved into place.

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(a) In determining $V(x)$ on the x-axis, first consider the region $x > a$, to the right of both charges. Then the distance from the negative charge to the point being considered is $x - a$, and the distance from the positive charge is $x + a$. The potential is then

$$V(x) = \frac{-kQ}{(x-a)} + \frac{kQ}{(x+a)}$$

For values of x between $-a$ and $+a$, the expression for the distance to the negative charge becomes $a - x$ for the negative charge, and the potential then becomes

$$V(x) = \frac{-kQ}{(a-x)} + \frac{kQ}{(x+a)}$$

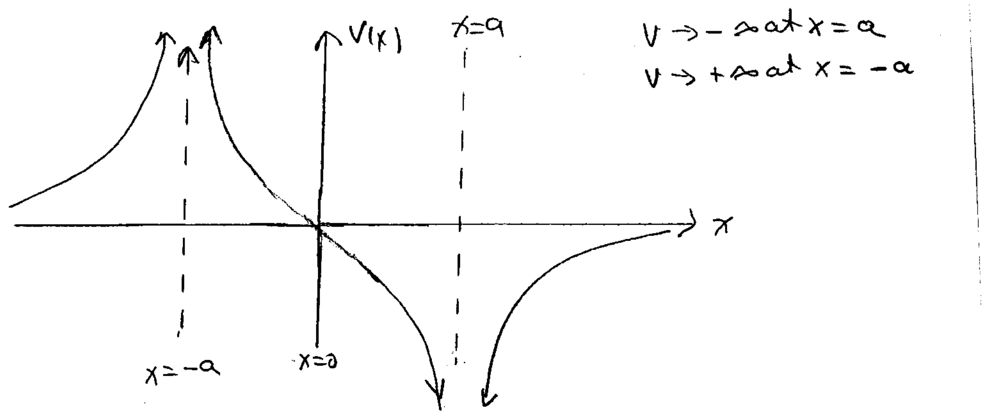
For $x < -a$, the distance to the positive charge becomes $-(x+a)$. Watch the signs, remember x carries a negative sign in this region, so that $-(x + a)$ is a positive number for $x < -a$. Now $V(x)$ becomes

$$V(x) = \frac{-kQ}{(x-a)} + \frac{kQ}{-(x+a)}$$

We can combine these three cases in one expression by using absolute values:

$$V(x) = \frac{-kQ}{|x-a|} + \frac{kQ}{|x+a|}$$

Sketch :



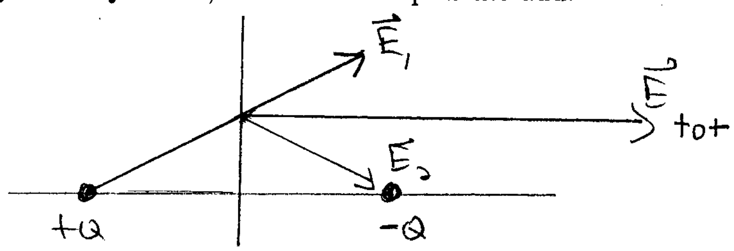
(b) $V(x) = 0$ at $x = 0$ as well as $\pm\infty$. The electric field at $x = 0$ is not zero: $\vec{E} = \frac{2kQ}{a^2}\hat{\imath}$, in the positive x direction. At $\pm\infty$, the electric field and potential are both zero.

(c) Consider any point on the y-axis. The distance from that point to either charge is the same ($\sqrt{a^2 + y^2}$). Since the charges are opposite, $V(y) = 0$ for *all* points on the y-axis! But that does not mean the electric field is zero there!

(d) The work done on a particle in an electrostatic potential is independent of the path, so we can consider a path that comes in along the positive y-axis. From (c) we know that the potential is zero everywhere on the y-axis, so we would do *no work* in moving the third charge $+Q$ from infinity to any point on the y-axis.

(e) Even though the potential is zero on the y-axis, the electric field is not zero there. By the symmetry (see sketch below) the y-components of the field from $+Q$ and $-Q$ cancel, and the x-components add.

$\vec{E} = E_x = \frac{2kQ}{2a^2}\hat{i}$



How does this make sense? We have done no work to move the charge from infinity to a point on the y-axis, and yet the electric field is not zero there. Recall the definition of work: $W = \int \vec{F} \cdot \vec{dl}$. The electric field is in the $+x$ direction, and we are moving the charge along the y-axis, so $\vec{dl}$ is in the y direction, and the dot product vanishes. So, although there is an electric field, we are moving the particle perpendicular to it, and therefor are doing no work.

If we moved the particle along a different path which was not always perpendicular to $\vec{E}$, sometimes we would do positive work, sometimes negative work, but the sum would always be zero. The path-independence of the electrostatic force guarantees this.

(f) Before the third particle is moved into place the electrostatic potential energy is

$U = \frac{-kQ^2}{2a}$

After the third particle is moved into place we need to add two new terms, one for the potential energy between the new charge and each of the original ones.

$U = \frac{-kQ^2}{2a} + \frac{-kQ^2}{\sqrt{2}a} + \frac{kQ^2}{\sqrt{2}a}$

But these two new terms cancel, which they must if the answer to (d) is correct.

II. A spherical shell of *nonconducting* material has an inner radius a and outer radius b. It carries a uniform *volume* charge distribution ρ. Take the zero of the electrostatic potential to be zero at infinity.

(a) Determine the total charge Q contained in the spherical shell.
(b) Determine the electrostatic potential $V(r)$ as a function of r for $r > b$.
(c) Determine the electric field $\vec{E}(r)$ for the region $a < r < b$. Using this result for $\vec{E}$, determine the potential $V(r)$ for the region $a < r < b$.
(d) Determine the potential $V(r)$ for the region $r < a$.
(e) Sketch $V(r)$ for all r and indicate any points or regions where the electric field $\vec{E}$ is zero.
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(a) To determine the total charge, integrate the charge density ρ over the volume. The only tricky part is that this is a spherical shell not a solid sphere.

$Q = \int_a^b \rho 4\pi r^2 dr$

$Q = 4\pi\rho(b^3 - a^3)/3$

(b) For $r > b$, the total charge in enclosed, so the electric field and potentials are the same as that for a point charge Q at the origin.

$V(r) = \frac{kQ}{r} = \frac{Q}{4\pi\epsilon_o r}$ or in terms of ρ $V(r) = \frac{\rho(b^3-a^3)}{3\epsilon_o r}$

(c) Use Gauss' Law to determine the electric field in the region $a < r < b$.

$\int \vec{E}\cdot d\vec{A} = Q_{encl}/\epsilon_o$

$4\pi r^2 E_r = 4\pi\rho(r^3 - a^3)/3\epsilon_o$

$E_r = \frac{\rho(r^3-a^3)}{3\epsilon_o r^2}$ radially outward.

We can determine the potential inside the shell by integrating the potential from b into a value of $r > a$. This gives us ΔV, which we than add to the potential at $r = b$.

$\Delta V = -\int \vec{E}\cdot d\vec{l} = +\int E_r dr$

where the change in sign comes about because $\vec{E}$ and $d\vec{l}$ are antiparallel–$\vec{E}$ points outward and $d\vec{r}$ points inward since we are integrating inward from $b \to r$.

$\Delta V = \frac{\rho}{3\epsilon_o}\int_r^b (r - \frac{a^3}{r^2})dr = \frac{\rho}{3\epsilon_o}(\frac{b^2}{2} - \frac{r^2}{2} + \frac{a^3}{b} - \frac{a^3}{r})$

This is the change in potential, which is added to $V(r = b)$ from part (b) to determine $V(r)$ for $a < r < b$.

$V(a < r < b) = V(b) + \Delta V = \frac{\rho}{3\epsilon_o}(\frac{3b^2}{2} - \frac{r^2}{2} - \frac{a^3}{r})$

(d) Since the electric field is zero for $r < a$, the potential is constant at the value $V(r = a)$.

$V(r < a) = \frac{\rho}{3\epsilon_o}(\frac{3b^2}{2} - \frac{3a^2}{2})$

(e) Sketch is below, the electric field is zero in regions where the derivative of V is zero, which is the case for $r < a$. The electric field is also zero at $r = \infty$.

(f) The kinetic energy will equal the potential energy lost, $\Delta U = KE = kQq(\frac{1}{2b} - \frac{1}{10b})$