Physics 102 – Pledged Problem 3

Time allowed: **2 hours at a single sitting**

DUE 4PM MONDAY, February 4, 2008, in the boxes marked Phys 101-102 in the physics lounge. You may use your own textbook, your notes, and a non-programmed calculator. You may also consult the on-line solutions to the corresponding suggested problems. You should consult no other help. Show how you arrived at your answer; the correct answer by itself may not be sufficient.

Further instructions:

(a) Write legibly on **one** side of 8.5” x 11” white or lightly tinted paper.
(b) Staple all sheets together, including this one, in the upper left corner and make one vertical fold.
(c) On the outside, staple side up. Print your name in capital letters, your LAST NAME first followed by your FIRST NAME.
(d) Below your name, print the phrase “Pledged Problem 3”, followed by the due date.
(e) Also indicate **start time** and **end time.**
(f) Write and sign the pledge, with the understanding you may consult the materials noted above.

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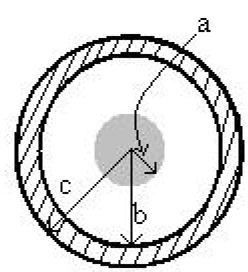
1. Consider a spherical cloud of charge of radius a, with a non-uniform charge density, i.e. charge per unit volume, given by $\rho = {}^{A}/_{r^2}$, where *A* is a constant and *r* the radial distance from the center. It is surrounded by a concentric conducting shell of inner radius *b* and outer radius *c* which carries a net charge – q, whose magnitude is greater than the total charge on the inner cloud.

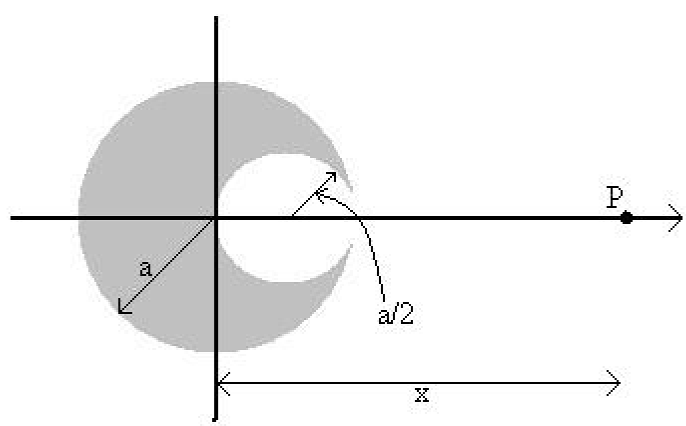
   a) Use Gauss’ Law to determine the electric field at all points in space. Indicate the Gaussian surfaces used.

   b) Show on a carefully labeled figure the radial dependence of the electric field.

   c) What are the charge densities, σ_b and σ_c , on the inner and outer faces of the conductor?



2. Consider a spherical cloud of charge of uniform charge density ρ and radius *a*, containing a spherical cavity of radius ${}^{a}/_{2}$, as shown in the figure. Calculate the electric field at point P a distance x from the center of the large cloud.



Solution – Pledged Problems #3

1. (a) This problem involves applying Gauss' Law. Because the charge distributions have radial symmetry (the charge densities for both the cloud and conducting shell depend only on r, the radial distance from the center), then one will use Gaussian surfaces having the same symmetry, namely concentric spheres of varying *r*. Applying Gauss' Law first to a sphere inside the cloud, with radius *r* < a, one gets

$$\oint \vec{E}\cdot \mathrm{d}\vec{A} = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_0^{\mathrm{r}} \rho(\mathrm{r})4\pi \mathrm{r'}^2 \mathrm{dr'} = \frac{1}{\epsilon_0}\int_0^{\mathrm{r}} \frac{\mathrm{A}}{\mathrm{r'}^2} 4\pi \mathrm{r'}^2 \mathrm{dr'} = \frac{4\pi \mathrm{A}}{\epsilon_0}\int_0^{\mathrm{r}} \mathrm{dr'} = \frac{4\pi \mathrm{Ar}}{\epsilon_0}$$

Again, symmetry indicates that on the Gaussian surface, the electric field, $\vec{E},$ will be constant and parallel to $d\vec{A}$ and so can be removed from the surface integration. The surface area is simply $4\pi r^2$, so that, for $r < a,$

$$\vec{E} = \frac{A}{r\varepsilon_0}\hat{r} \quad (r < a)$$

For $a \le r < b$, the charge enclosed will be the total charge in the inner cloud, or $Q_{enc} = 4\pi Aa$, and so Gauss' Law indicates that

$$\oint \vec{E}\cdot \mathrm{d}\vec{A} = \mathrm{E}4\pi\mathrm{r}^2 = \frac{Q_{encl}}{\epsilon_0} = \frac{4\pi Aa}{\epsilon_0}$$

and, therefore,

$$\vec{E} = \frac{Aa}{r^2\varepsilon_0}\hat{r} \quad (a \le r < b)$$

Within the conducting shell, for $b \le r \le c$, sufficient charge will accumulate on the inner radius (equal in magnitude to the total charge in the cloud but of opposite sign), so that the field inside the conductor will be zero,

$$\vec{E} = 0 \quad (b \le r \le c)$$

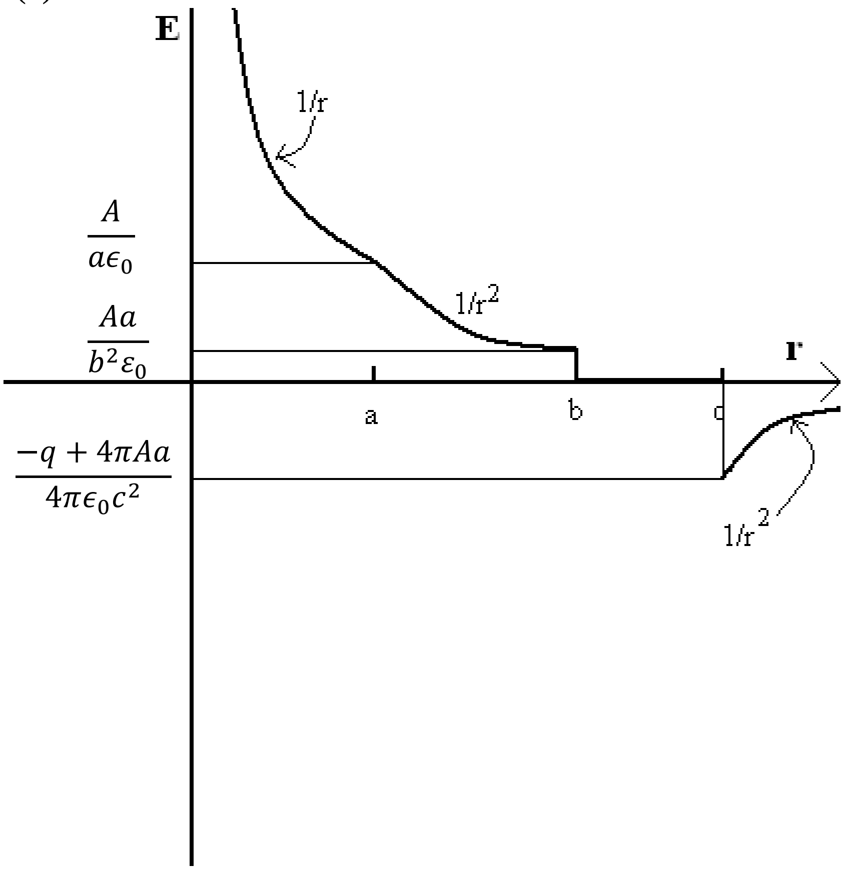
Outside the conducting shell, for $c < r$,

$$\oint \vec{E}\cdot \mathrm{d}\vec{A} = \mathrm{E}4\pi\mathrm{r}^2 = \frac{Q_{encl}}{\epsilon_0} = \frac{-q + 4\pi Aa}{\epsilon_0}$$

and, therefore,

$$\vec{E} = \frac{-q + 4\pi Aa}{4\pi\varepsilon_0 r^2}\hat{r} \quad (c < r)$$

(b)



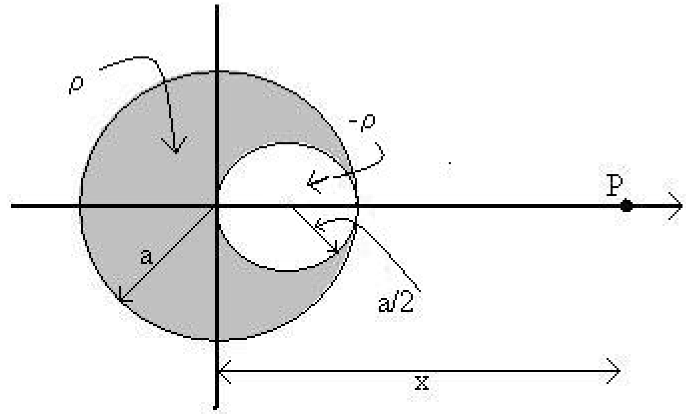
(c) The charge density on the inner face of the conductor, σ_b, will need to be sufficient to produce a total charge equal to in magnitude, but opposite in sign, to the charge in the cloud. This charge density will be uniform over the inner face. As determined in part (a), the total charge in the cloud is $4\pi Aa$ *C*, and the total surface area on the inner face is $4\pi b^2$ m^2. Therefore, the charge density on the inner face will be

$$\sigma_b = -\frac{4\pi Aa}{4\pi b^2}\frac{C}{m^2} = -\frac{Aa}{b^2}\frac{C}{m^2}$$

Similarly, the charge density on the outer face of the conductor, σ_c, will be the remaining charge on the conductor (after subtracting the charge distributed on the inner face from the total charge, -q, on the shell), divided by the total surface area of that outer face, $4\pi c^2$ m^2. Therefore, the charge density on the outer face will be

$$\sigma_c = -\frac{q - 4\pi Aa}{4\pi c^2}\frac{C}{m^2}$$

2. This problem involves the Law of Superposition of electric fields. The key to solving the problem is to recognize that the net charge density shown in the figure can be treated as associated with two spherical clouds of uniform charge. The first, having a uniform charge density of ρ and radius a, is centered at the origin; the second, having a uniform charge density of -ρ and radius a/2, is centered at x = a/2. (When the charged densities in the cavity are added, they produce a net charge density of 0 in that volume).



The electric field associated with either charge distribution at a given distance from the center can be determined by using Gauss' Law or can be recalled from the example in the book. It is directed radially from the center and equals $Q_{tot}/4\pi\varepsilon_0 r^2$, where Q_{tot} is the total charge in the cloud and r is measured from the center of the sphere (the same electric field obtained by considering the entire charge to be located at the center of the cloud).

The electric field at P associated with the charge density in the large cloud will then simply be

$$\vec{E}_1 = \frac{Q_{tot}}{4\pi\varepsilon_0 r^2}\hat{x} = \frac{{}^4\!/_3\,\pi a^3\rho}{4\pi\varepsilon_0 x^2}\hat{x} = \frac{a^3\rho}{3\varepsilon_0 x^2}\hat{x}$$

The electric field at P associated with the charge density in the smaller cloud will be

$$\vec{E}_2 = \frac{Q_{tot}}{4\pi\varepsilon_0 r^2}\hat{x} = -\frac{{}^4\!/_3\,\pi({}^a\!/_2)^3\rho}{4\pi\varepsilon_0 (x - {}^a\!/_2)^2}\hat{x} = -\frac{a^3\rho}{24\varepsilon_0 (x - {}^a\!/_2)^2}\hat{x}$$

The total field will be the sum of these fields –

$$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 = \left(\frac{a^3\rho}{3\varepsilon_0 x^2} - \frac{a^3\rho}{24\varepsilon_0 (x - {}^a\!/_2)^2}\right)\hat{x} = \frac{a^3\rho}{3\varepsilon_0}\left(\frac{1}{x^2} - \frac{1}{8(x - {}^a\!/_2)^2}\right)\hat{x}$$