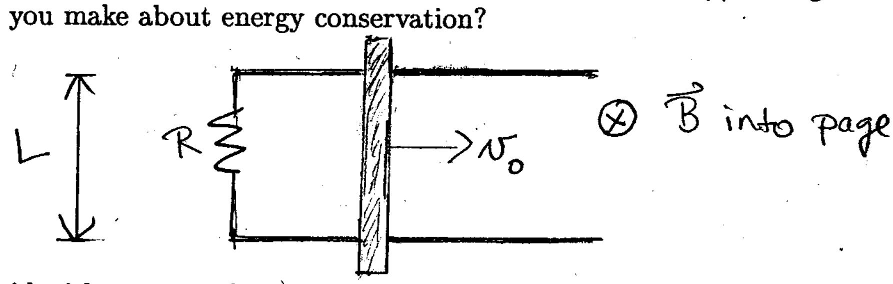
Physics 102– Pledged Problem 10

Time allowed: **2 hours at a single sitting**
Due 5PM Monday, April 17, 2006, in the boxes marked Phys 101-102 in the physics lounge. You may use your own textbook, your notes, and a non-programmed calculator. You may also consult the on-line solutions to the corresponding suggested problems. You should consult no other help. Show how you arrived at your answer; the correct answer by itself may not be sufficient.
Further instructions:
(a) Write legibly on **one** side of 8.5" x 11" white or lightly tinted paper.
(b) Staple all sheets together, including this one, in the upper left corner and make one vertical fold.
(c) On the outside, staple side up, print your name in capital letters, your LAST NAME first followed by your FIRST NAME.
(d) Below your name, print the phrase "Pledged Problem 10", followed by the due date.
(e) Also indicate **start time** and **end time**.
(f) Write and sign the pledge, with the understanding that you may consult the materials noted above.

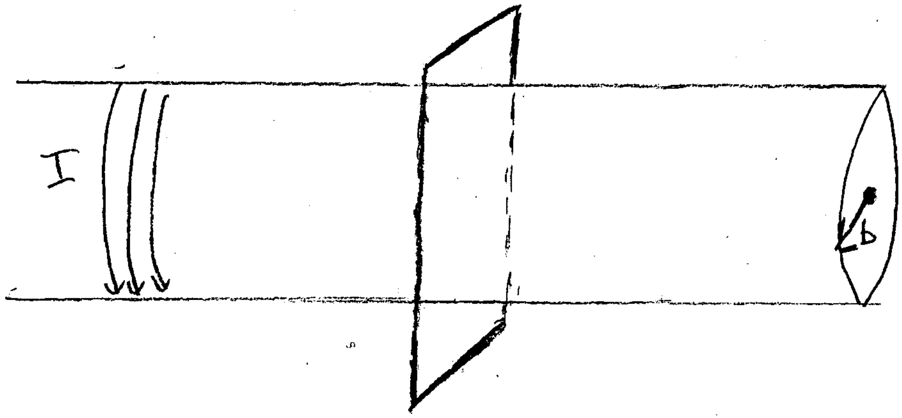
I. A conducting rod of mass m and negligable resistance is free to slide without friction on two horizontal, parallel rails, also of negligible resistance. The rails are separated by a distance L and are connected together by a resistor R to make a circuit. The entire circuit is in a uniform magnetic field $\vec{B}$ directed into the page. At $t = 0$ the rod is given an initial velocity v_o in the $+x$ direction as shown.

(a) Determine the current (both direction and magnitude) through the rod when it has velocity v.
(b) Determine the force (both direction and magnitude) on the rod when it has velocity v.
(c) Determine the velocity of the rod as a function of time $v(t)$. Hint: Write $F = m\frac{dv}{dt}$.
(d) Determine the current through the circuit as a function of time $I(t)$.
(e) Determine the power dissipated in the resistor as a function of time $P(t)$. Integrate $P(t)$ from $t = 0 \to \infty$. What statement can you make about energy conservation?



II. A very long solenoid with n turns of wire per unit length and radius b carries a current $I(t)$ which decays with time as $I(t) = I_o e^{-t/\tau}$. The direction of the current in the solenoid is as shown below. The solenoid passes through a square coil of wire of side slightly larger than 2b and resistance R.

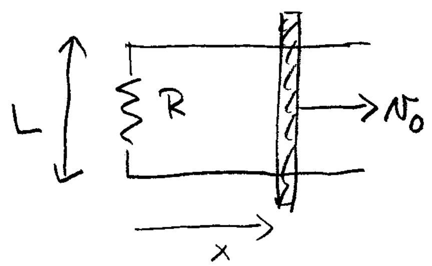
(a) Determine the electric field both inside and outside the solenoid as the current is decaying as given above. Be sure to indicate both the direction and magnitude of the field.
(b)Determine the current through the coil as a function of time $I_c(t)$ as the solenoid current decays.
(c) Determine the total energy dissipated in the resistance of the square coil from $t = 0 \to \infty$. Where does this energy come from?



Phys 102

Pledged Problem 10

I.



$\otimes$ $\vec{B}$ into page

(a) When the rod is a distance x from the resistor,

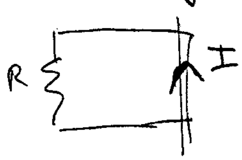
Area = Lx

Flux = $\int \vec{B} \cdot d\vec{A} = BLx = \phi_B$

$$\left| \frac{d\phi_B}{dt} \right| = BLv = \mathcal{E}$$

$I = \dfrac{\mathcal{E}}{R} = \dfrac{BLv}{R}$ with the direction given by Lenz's Law

ϕ_B is changing downward, so the current will be in the direction to produce a $\vec{B}$ field upward through the loop:



$$\boxed{I = \frac{BLv}{R} \quad \text{Counterclockwise}}$$

(b) The force on a current-carrying conductor is

$$F = I\vec{L} \times \vec{B} = \frac{BLv}{R} \cdot LB$$

to the left, or in the $-x$ direction

$$\boxed{\vec{F} = -\frac{B^2L^2v}{R}\,\hat{i}}$$

(c) $$F = m\frac{dv}{dt} = -\frac{B^2L^2v}{R}$$

$$\frac{dv}{v} = \frac{-B^2L^2}{mR}\,dt$$ Integrate

$$\ln v = \frac{-B^2L^2t}{mR} + A$$

$\hookrightarrow$ constant of integration

Exponentiate:

$$v(t) = C e^{-B^2L^2t/mR}$$ C = also a constant.

Determine C from the condition $v(t=0) = v_0$

$$\boxed{v(t) = v_0 e^{-B^2L^2t/mR}}$$

(d) In (a) we saw that

$$I = \frac{BLv}{R} \quad \text{so}$$

$$\boxed{I(t) = \frac{BLv_0}{R} e^{-B^2L^2t/mR}}$$

(e) $P(t) = I^2(t)R$

$$\boxed{P(t) = \frac{B^2L^2v_0^2}{R} e^{-2B^2L^2t/mR}}$$

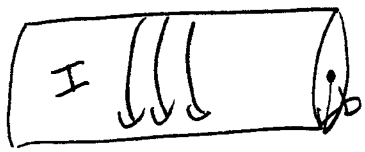
$$\int_0^\infty P(t)\,dt = \frac{B^2L^2v_0^2}{R} \int_0^\infty e^{-2B^2L^2t/mR}\,dt$$

$$= \frac{\cancel{B^2}\cancel{L^2}v_0^2}{\cancel{R}} \cdot \frac{-m\cancel{R}}{2\cancel{B^2}\cancel{L^2}} \underbrace{e^{-2B^2L^2t/mR}\Big|_0^\infty}_{(0-1)=-1}$$

$$\boxed{\int_0^\infty P(t)\,dt = \frac{mv_0^2}{2}}$$

The initial kinetic energy of the rod appears as heat in the resistor!

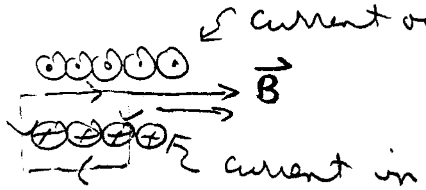
II:



$I = I_0 e^{-t/\tau}$

(a) The changing magnetic field in the solenoid induces an electric field. In a solenoid,

$B = \mu_0 n I$ by Ampere's law:



B = 0 outside
B = constant inside

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$

$B l = \mu_0 n I l$

$n = \#$ of turns/length

Then $B(t) = \mu_0 n I_0 e^{-t/\tau}$

By symmetry the E field must form concentric loops. By Faraday's law we have

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt}$$

for $r < b$, $\phi_B = \pi r^2 B = \pi r^2 \mu_0 n I_0 e^{-t/\tau}$

$$\frac{d\phi_B}{dt} = \frac{-\pi r^2 \mu_0 n I_0}{\tau} e^{-t/\tau}$$

Even though field $\vec{B}$ is up out of the page, the flux is decreasing, since I is decreasing with time. The direction of the electric field will be such as to maintain the flux — so when viewing from the right, I and $\vec{E}$ are both counterclockwise.

$$\oint \vec{E} \cdot dl = -\frac{d\phi_B}{dt}$$

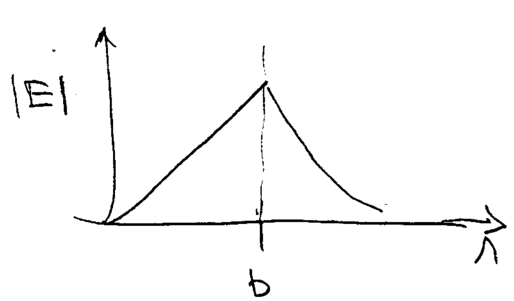
$$|2\pi r E| = \frac{\pi r^2 \mu_0 n I_0}{\tau} e^{-t/\tau}$$

$$\boxed{|\vec{E}| = \frac{\mu_0 n I_0 r e^{-t/\tau}}{2\tau} \quad \text{ccw}} \qquad r < b$$

for $r > b$, the flux enclosed extends only out to $r = b$

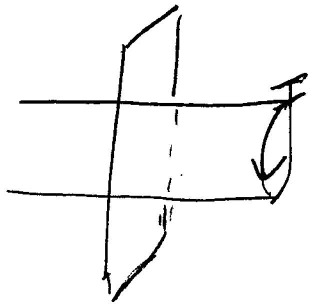
$$\oint \vec{E} \cdot d\vec{l} = 2\pi r E = \frac{\mu_0 n \pi b^2 I_0 e^{-t/\tau}}{\tau}$$

$$\boxed{|\vec{E}| = \frac{\mu_0 n b^2 I_0 e^{-t/\tau}}{2 r \tau} \qquad \begin{array}{c} r > b \\ \text{direction still ccw} \end{array}}$$



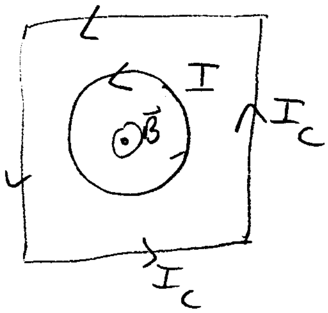
|E| increases linearly out to $r = b$ & falls as $1/r$ for $r > b$.

(b).



The Solenoid passes through the coil. The dimensions of the coil don't matter, since the changing flux is confined to the solenoid.

$$\mathcal{E} = -\frac{d\phi_B}{dt} = \frac{-\mu_0 n I_0 e^{-t/\tau}}{\tau}\overbrace{(\pi b^2)}^{\text{area of solenoid}}$$



Again, the direction of the induced current will be ccw as viewed from the right. As $\vec{B}$ decreases, the induced current opposes the change.

$$\boxed{I_c = \frac{\mathcal{E}}{R} = \frac{\mu_0 n I_0 \pi b^2 e^{-t/\tau}}{\tau R} \quad \text{ccw}}$$

(c) $$P = I^2 R = \frac{(\mu_0 n I_0 \pi b^2)^2 e^{-2t/\tau}}{\tau^2 R^{\cancel{2}}} \cdot \cancel{R}$$

Integrate $0 \to \infty$ to get total energy dissipated

$$\int_0^\infty P(t)\,dt = \frac{(\mu_0 n I_0 \pi b^2)^2}{\tau^2 R}\int_0^\infty e^{-2t/\tau}\,dt$$

$$= \frac{\mu_0^2 n^2 I_0^2 \pi^2 b^4}{\tau^2 R}\left(\frac{-\tau}{2}\right)\underbrace{e^{-2t/\tau}\Big|_0^\infty}_{-1}$$

Total energy dissipated

$$\boxed{E_{TOT} = \frac{\mu_0^2 n^2 I_0^2 \pi^2 b^4}{2\tau R}}$$

This energy comes from the energy initially stored in the magnetic field.

$B = \mu_0 I / 2\pi r$

$\mu_0 \sim \frac{B \cdot r}{I} = \frac{T\,m}{A}$

$\Omega \sim \frac{V}{A}$

$[T] = \frac{N\,s}{C\,m}$

Check units:

$$[E_{TOT}] = \frac{\left(\frac{T-m}{A}\right)^2 \cdot \frac{1}{m^2} \cdot A^2\, m^4}{\sec \cdot \text{Volt}/A}$$

$$= \frac{T^2 C^2 m^4}{s\, J\, s}$$

$$= \frac{N^2 \cancel{s^2}}{C^2 m^2} \cdot \frac{\cancel{C^2} m^{\cancel{4}\,2}}{J\, \cancel{s^2}}$$

$$\sim \frac{N^2 - m^2}{N - m} \sim N-m = \text{Joules!}$$

Hint on checking units: Use fundamental equations — for example, for $\vec{B}$, use

$$\vec{F} = q\, \vec{v} \times \vec{B}$$

$$[B] = \text{Tesla} = \frac{N-s}{C\,m}$$