

RLC Circuits

It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong.

Richard Feynman (1918-1988)

OBJECTIVES

To observe free and driven oscillations of an RLC circuit.

THEORY

The circuit of interest is shown in Fig. 1, including sine-wave sources. We start with the series connection, writing Kirchoff's law for the loop in terms of the charge q_C on the capacitor and the current $i = dq_C/dt$ in the loop. The sum of the voltages around the loop must be zero, so we obtain

$$v_L + v_R + v_C = v_D \sin(\omega t) \quad (1)$$

$$L \frac{d^2 q_C}{dt^2} + R \frac{dq_C}{dt} + \frac{q_C}{C} = v_D \sin(\omega t) \quad (2)$$

For reasons that will become clear shortly, we rewrite this as

$$\frac{d^2 q_C}{dt^2} + \frac{2}{\tau} \frac{dq_C}{dt} + \omega_0^2 q_C = \frac{v_D}{L} \sin(\omega t) \quad (3)$$

with $\tau = 2L/R$ and $\omega_0^2 = 1/LC$. Two situations will be of interest. We will examine the free oscillations, when v_D is exactly zero, meaning that the sine-wave generator has been replaced with a short circuit. We will then connect the sine wave generator and calculate the response as a function of frequency.

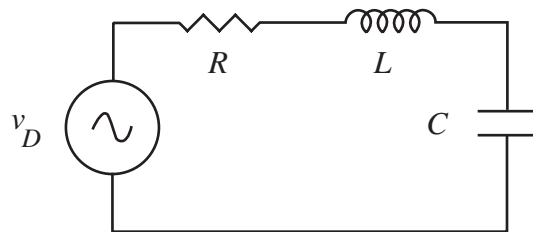


Fig. 1 Idealized series RLC circuit driven by a sine-wave voltage source.

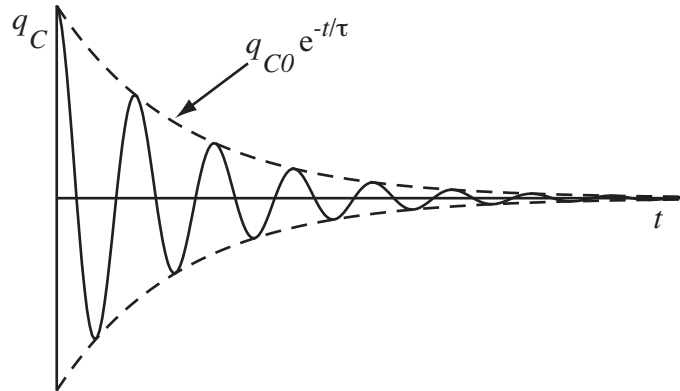


Fig. 2 Damped oscillation, showing decay envelope

If the resistance in the circuit is small, the free oscillations are of the form

$$q_C = q_{C0} e^{-t/\tau} \cos(\omega_1 t + \phi) \quad (4)$$

Where q_{C0} and ϕ are determined by initial conditions, and

$$\omega_1 = \omega_0 [1 - (\omega_0 \tau)^{-2}]^{1/2} \quad (5)$$

This solution is plotted in Fig. 2 for a case where the capacitor is initially charged and no current is flowing. (For a mass on a spring the equivalent situation would be to pull the mass aside and release it from rest.) Evidently there are oscillations at ω_1 , approximately equal to ω_0 , within an exponential envelope. Note that the amplitude falls to $1/e$ of the initial value when $t = \tau$.

As τ gets smaller (larger resistance R), ω_1 becomes smaller and finally imaginary. The corresponding solutions do not oscillate at all. For $\omega_0 < 1/\tau$, there are two exponentials

$$q_C = A_1 e^{-t/\tau_1} + A_2 e^{-t/\tau_2} \quad (6)$$

where τ_1 and τ_2 differ somewhat from τ . When $\omega_0 = 1/\tau$ the solution is slightly simpler:

$$q_C = (A_1 + A_2 t) e^{-t/\tau} \quad (7)$$

If the capacitor is initially charged, these results tell us we will get a monotonic decay for sufficiently large R . The case $\omega_0 = 1/\tau$ is referred to as "critical damping" because the charge reaches zero in the shortest time without changing polarity.

The solution for sine-wave driving describes a steady oscillation at the frequency of the driving voltage:

$$q_C = A \sin(\omega t + \phi) \quad (8)$$

We can find A and ϕ by substituting into the differential equation and solving:

$$A = \frac{v_D / L}{\left[(\omega_0^2 - \omega^2)^2 + (2\omega / \tau)^2 \right]^{1/2}} \quad (9)$$

$$\tan \phi = \frac{2\omega / \tau}{(\omega^2 - \omega_0^2)} \quad (10)$$

These two equations are plotted in Fig. 3, where we use the fact that $v_C = q_C / C$ to plot the voltage across the capacitor relative to the driving voltage. (Our apparatus does not allow us to observe the phase of the response, so we won't consider that further.)

The angular frequency for maximum amplitude is given by

$$\omega_{peak} = \omega_0 \left[1 - 2 / (\omega_0 \tau)^2 \right]^{1/2} \quad (11)$$

At the maximum, the oscillation amplitude is considerably greater than the driving amplitude. In fact, if τ were infinite (no damping), the response would be infinite at ω_0 .

Since the shape of the peak in v_C characterizes the resonance, it is convenient to have some parameter to specify the sharpness of the peak. Traditionally, this is taken to be the full width $\Delta\omega$, shown in Fig. 3, at which the voltage or current have fallen to $1/\sqrt{2}$ of their peak

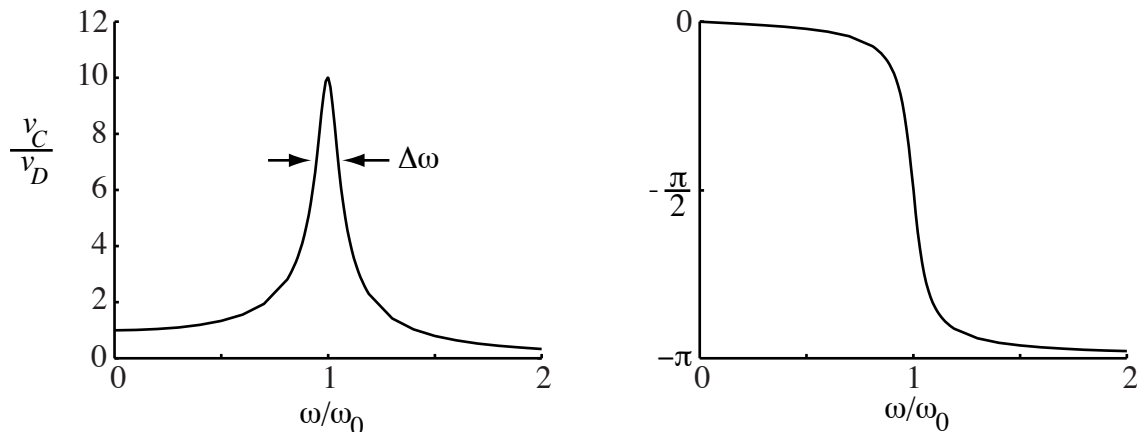


Fig. 3 Amplitude and phase of capacitor voltage as a function of frequency.

value. The reason for this choice is that the power dissipation is proportional to i^2 , so these frequencies correspond to the points at which the power dissipation is half of the maximum. Using Eq. 9 we find that the width is related to τ by

$$\Delta\omega = \frac{2}{\tau} \quad (12)$$

demonstrating the intimate relation between time and frequency response parameters. In obtaining this result, we assumed $\omega_0\tau \gg 1$, so that $\omega \approx \omega_0$ at the peak.

EXPERIMENTAL PROCEDURE

The RLC circuit is assembled from a large solenoid, a capacitor on the circuit board, and an additional variable resistance to change the damping. The circuit can be charged up with a DC power supply to study the free oscillations, or driven with a sine wave source for forced oscillations.

Free oscillations

To study the homogeneous solution we will use LoggerPro to record the voltage across the capacitor as a function of time. The required circuit is shown in Fig. 4. Since the coil is not made with superconducting wire, we account for the wire resistance with R_L . The variable resistor, shown by a resistor symbol with an arrow in the middle, should be set to minimum resistance initially. (You can check with the DMM in ohmmeter mode, before connecting R into the circuit.) The power supply can be set to maximum output.

Start LoggerPro with the file RLC.cmb1. This will configure the program to collect data at 10,000 samples per second, triggered when the voltage decreases across 9.5 V. Set the switch to position b to charge the capacitor, and then start data acquisition. Flipping the switch to position

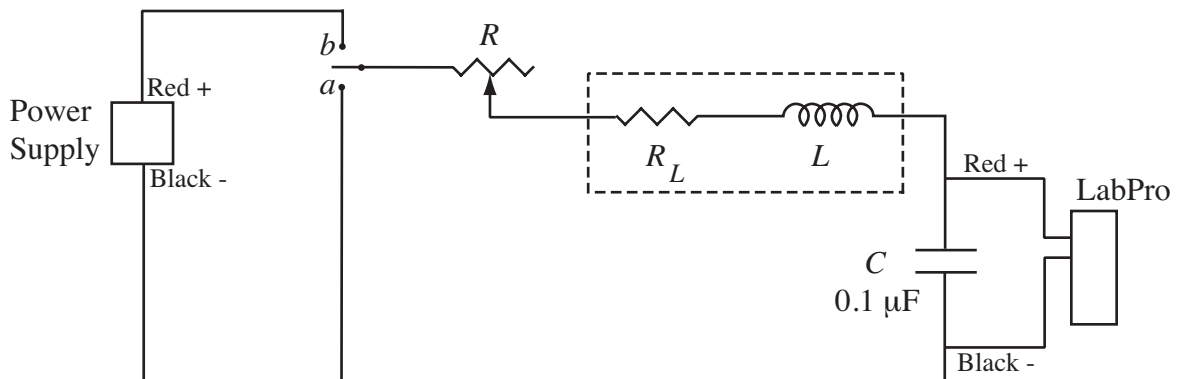


Fig. 4 Circuit for measuring free oscillations in an RLC circuit.

a will start the discharge, and should result in a plot resembling Fig. 2. There may be some irregularity at the beginning, due to bouncing when the switch first makes contact, but you can ignore that in your analysis.

When you have a reasonable looking plot, select the portion of the data after any effects of switch bounce, and fit it to a damped sine wave. You should be able to obtain estimates of the time constant τ and the angular frequency ω for your circuit. (Delete the fit or restart RLC.cml before taking more data. Otherwise you will have to wait while the program tries to fit the new data.)

Now increase the variable resistor R slightly and obtain another voltage vs time plot. Describe what happens to the time constant and to the shape of the curve as R becomes progressively larger.

Critical damping occurs when R is just big enough that the voltage no longer crosses zero during the decay. Try to find this setting, and measure the value of the variable resistor using a DMM. The exact value is not very clear, but make a reasonable attempt. Reminder: You must disconnect R from the circuit to get an accurate measurement with the DMM.

Driven oscillations

To study the driven solution we need to provide a sine-wave signal source, as shown in Fig. 5. Our source is equivalent to an ideal sine-wave generator in series with a $50\ \Omega$ resistor, labeled R_g in the diagram. Since R_g is large enough to affect the damping of the circuit, we add the $3.3\ \Omega$ resistor in parallel to reduce the effective resistance of the generator. The other change from Fig. 4 is to replace the LabPro with a DMM, set to read AC voltage. (The DMM can read AC voltages between about 30 Hz and 1000 Hz. It is unreliable outside of that range.)

Connect the circuit as shown in Fig 5, and then vary the driving frequency to find the frequency at which v_C is maximum. This identifies the resonance frequency, in Hertz. Next, you should plot v_C as a function of frequency, taking care to get enough data around the resonance frequency to clearly define the curve. This goes very quickly if you enter f and v_C directly into

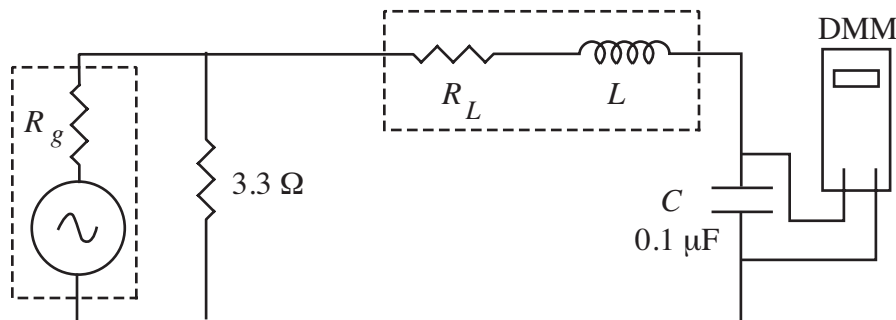


Fig. 5 Circuit for measuring driven oscillations in an RLC circuit.

Graph.cmbl and then pick data points to trace the regions of interest. Does your plot look like Fig. 3?

You can get an accurate measure of the width by finding the frequencies just above and just below the resonance at which v_C is reduced by $1/\sqrt{2}$ from the peak value. The width $\Delta\omega$ is then $2\pi\Delta f$, where the 2π converts from frequency in Hertz to angular frequency. Do your measured values of $\Delta\omega$ and τ satisfy Eq. 12?