Physics 102 Spring 2007: Worked Example - Ampère's Law.

An elaboration of the example worked out during class.

1. A long, straight wire of radius R carries a steady current I that is uniformly distributed over the circular cross section of the wire. Find the magnetic field both outside and inside the wire. A sketch of the wire is shown in Fig. 1.

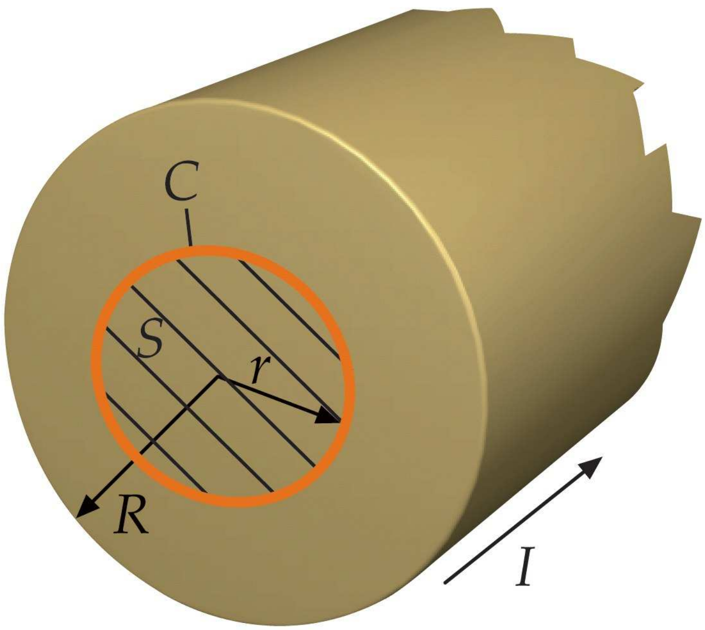

Figure 1: Worked Problem

Solution

We can use Ampère's law to calculate $\vec{B}$ because of the *high degree of symmetry.* At a distance r (Fig. 1), we know from the Biot-Savart law that $\vec{B}$ is tangent to the circle of radius r about the wire and $\vec{B}$ is constant in magnitude everywhere on the circle. The current through the surface S bounded by C depends on whether r is less than or greater than the radius of the wire R.

Ampère's law is used to relate the line integral of $\vec{B}$ around the curve C to the current enclosed by the surface S bounded by C

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_C. \tag{1}$$

Evaluating the line integral of $\vec{B}$ around a circle of radius r that is concentric with the wire, yields

$$\oint_C \vec{B} \cdot d\vec{l} = \oint_C B\, dl = \overbrace{B \oint_C dl}^{\text{Usefulness of Ampère's Law}} = B\, 2\pi r. \tag{2}$$

Substituting this expression into Eq. 1 and solving for B

$$B\, 2\pi r = \mu_0\, I_C \tag{3}$$

$$B = \frac{\mu_0\, I_C}{2\,\pi\, r} \tag{4}$$

where I_C is the current bounded by C. This implies there are two regions which must be considered: region (1) outside the wire ($r > R$) and region (2) inside the wire ($r < R$). Outside the wire, $r > R$ so the total current passing through the surface bounded by C is

$$I_C = I \tag{5}$$

;therefore,

$$B = \frac{\mu_0\, I}{2\,\pi\, r} \quad (r \geq R). \tag{6}$$

Inside the wire, $r < R$ so one needs to consider the definition of current in terms of a current density.

$$I_C = \int_0^r \vec{J} \cdot d\vec{A} = \int_0^r J dA. \tag{7}$$

Note: $\vec{J} \cdot d\vec{A}$ ***is*** JdA ***since*** $\vec{J}$ ***is parallel (Dunning notation*** $\|^l$***) with*** $d\vec{A}$***.***

Since $dA = 2\,\pi\,r\,dr$ for this geometry, one obtains:

$$I_C = \int_0^r J2\,\pi\,r^{'}\,dr^{'} = 2\,\pi \int_0^r Jr^{'}\,dr^{'} \tag{8}$$

If J is uniform, then it may be treated as a constant inside the integral; therefore, yielding:

$$I_C = 2\,\pi\,J \int_0^r r^{'}\,dr^{'} = J\,\pi\,r^2 \tag{9}$$

With the assumption that J is uniform, we conclude that (from the definition of J):

$$J = \frac{I}{\pi\,R^2} \tag{10}$$

With this value of J, one obtains:

$$I_C = \frac{I}{\pi\,R^2}\,\pi\,r^2 = I\,\frac{r^2}{R^2} \tag{11}$$

If the current density ***is not*** uniform over the wire, then we are left with I_C given by Eq. 7:

$$I_C = 2\,\pi \int_0^r J(r^{'})r^{'}\,dr^{'} \tag{12}$$

Where $J(r^{'})$ is an explicit function for the current density and varies with the position within the wire.
Going back to the case of uniform current density:

$$I_C = I\,\frac{r^2}{R^2} \tag{13}$$

which implies that the magnetic field within the wire is given by:

$$B = \frac{\mu_0\,I_C}{2\,\pi\,r} = \frac{\mu_0\,I\,r}{2\,\pi\,R^2}\ \ (r \leq R). \tag{14}$$

The graph of the B field given below.

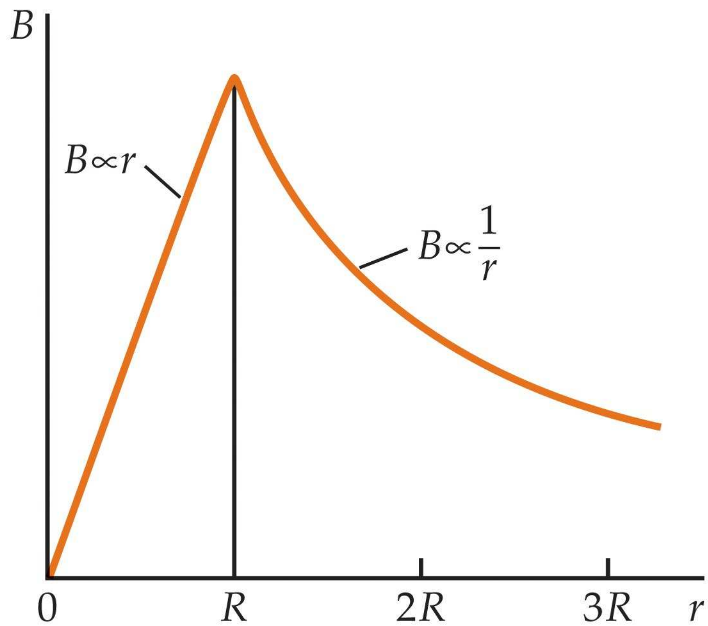

Figure 2: Worked Problem

Notice the linear behavior within the wire, and the peak value for the B field at $r = R$.