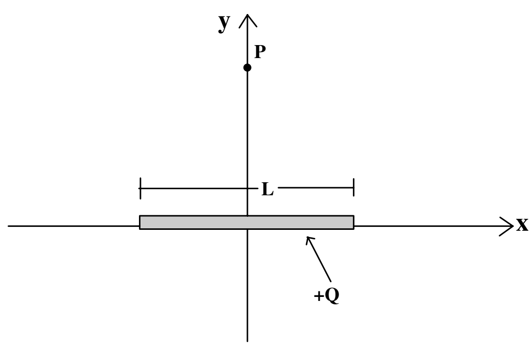
Physics 102 Spring 2006: Worked Example 1

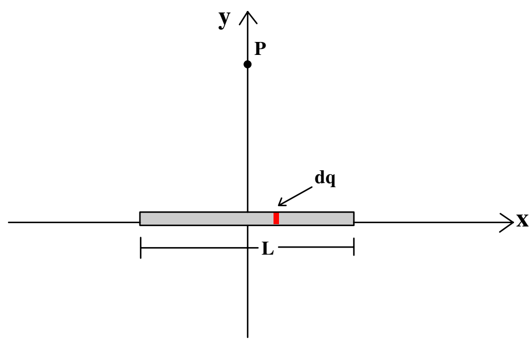
Please try to figure out the problem before referring to my solution

1. A wire of length L lies with it's central axis along the x-axis as shown in the figure below Fig. 1. A total charge +Q is placed uniformly on the wire (NOTE: this is another way of stating that the linear charge density is constant or $\lambda = \frac{+Q}{L}$). Calculate the electric field (vector) at a point P on the perpendicular bisector of the wire.



Solution

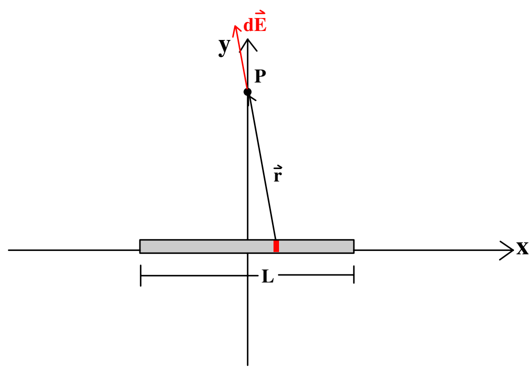
(I). Consider a small (extremely small) element of charge dq along the wire as shown in the figure below



The electric field ($d\vec{E}$) generated by this charge is given by Coulomb's law for electric fields:

$$d\vec{E} = \frac{k\,dq}{r^2}\hat{r} \tag{1}$$

The figure below shows $d\vec{E}$ due to the element of charge (dq).



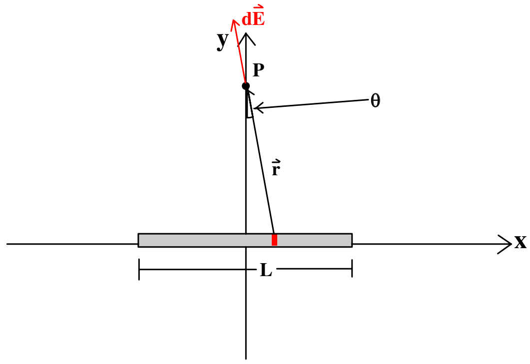
Due to the symmetry of the problem, the net electric field will point in the $+y$ direction. This simplifies Eq. 1 into a scalar integral.

(II). What is dq?

Since this is a one-dimensional problem:

$$dq = \lambda\,dx \tag{2}$$

With the angle defined in the figure below



(III). We may now begin to piece all our information together.

$$dE_y = \frac{k\,dq}{r^2}\cos\theta \tag{3}$$

$$dE_y = \frac{k\,\lambda\,dx}{r^2}\cos\theta \tag{4}$$

$$\Rightarrow E_y = \int_{\frac{-L}{2}}^{\frac{L}{2}} \frac{k\,\lambda\,dx}{r^2}\cos\theta \tag{5}$$

$$\Rightarrow E_y = 2\int_0^{\frac{L}{2}} \frac{k\,\lambda\,dx}{r^2}\cos\theta \tag{6}$$

$$\cos\theta = \frac{y}{r} \tag{7}$$

$$\Rightarrow E_y = 2\int_0^{\frac{L}{2}} \frac{k\,\lambda\,y\,dx}{r^3} \tag{8}$$

$$E_y = 2\,k\,y\,\lambda\int_0^{\frac{L}{2}} \frac{dx}{(y^2+x^2)^{\frac{3}{2}}} \tag{9}$$

Where the integral in eq. 9 is a standard integral (given in many textbooks including the one we use in class) with the solution given below

$$\int_0^{\frac{L}{2}} \frac{dx}{(y^2+x^2)^{\frac{3}{2}}} = \left(\frac{x}{y^2\sqrt{x^2+y^2}}\right)_{x=0}^{x=\frac{L}{2}} \tag{10}$$

$$\Rightarrow E_y = \frac{2\,k\,\lambda}{y}\left(\frac{\frac{L}{2}}{\sqrt{\left(\frac{L}{2}\right)^2+y^2}}\right) \tag{11}$$

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(IV). Check the limits!

When $y \gg L$, the wire should look like a point charge so let's verify this mathematically

$$E_y \backsimeq \frac{2\,k\,\lambda}{y}\left(\frac{\frac{L}{2}}{y}\right) \tag{12}$$

$$E_y \backsimeq \frac{k\,\lambda\,L}{y^2} \tag{13}$$

But $\lambda\,L = Q$ so for $y \gg L$:

$$E_y \backsimeq \frac{k\,Q}{y^2} \tag{14}$$

What about $y \ll L$? (This would be result for an infinite wire.)

$$E_y \backsimeq \frac{2\,k\,\lambda}{y} \tag{15}$$

We will prove the result for the infinite wire using another method - coming up in the next chapter.

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