Physics 102 Spring 2006: Maxwell's Equations - Differential Form

Most of this material is directly from Chapter 1 (sections 2 and 3) of "Introduction to Electrodynamics" by David J. Griffiths. I do not claim this work to be independent of this textbook. I just thought it would be a nice supplement for those in multi-variable calculus.

Since the electric and magnetic fields discussed in this course are three dimensional entities, it would be appropriate to discuss the behavior of these **vector** fields.

Let's write down what we know from this class:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \tag{1}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\left(\int \mathbf{B} \cdot d\mathbf{A}\right) \tag{2}$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0 \tag{3}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0\, I_{\text{enclosed}} + \mu_0\, \varepsilon_0 \frac{d}{dt}\left(\int \mathbf{E} \cdot d\mathbf{A}\right) \tag{4}$$

If the source of electric field is due to point charges or batteries (only), then Eq. 2 implies Kirchhoff's Loop rule:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

which is equivalent to stating that the electric field due to charges or batteries is "conservative".

Let's digress and discuss the ∇ operator, and the divergence and curl of vector fields.

The Operator ∇

The gradient has the formal appearance of a vector, ∇:

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The operator, ∇, is called **"del"** and is not a vector in the normal sense. We say that ∇ is a vector operator.

The Divergence

From the definition of ∇ we can construct the divergence:

$$\nabla \cdot \mathbf{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k})$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

NOTE: The divergence of a vector function $\mathbf{A}$ is itself a scalar.

Geometrical Interpretation: $\nabla \cdot \mathbf{A}$ is a measure of how much the vector $\mathbf{A}$ spreads out from the point in question. Imagine standing near a lake or pond. If you sprinkle some pollen on the water, and the pollen **spreads out** then you dropped it on a point of **positive divergence**. If the pollen **collects together**, then you dropped it on a point of **negative divergence**.

The Curl

From the definition of ∇ we construct the curl:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k})$$

$$= \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

NOTE: The curl of a vector function $\mathbf{A}$ is, like any cross product, a vector.

Geometrical Interpretation: $\nabla \times \mathbf{A}$ is a measure of how much the vector $\mathbf{A}$ "curls around" the point in question. Imagine standing near a lake or pond. If you sprinkle some pollen on the water, and the pollen **rotates** then you dropped it on a point of **nonzero curl**. A whirlpool would be a region of large curl.

Any conservative vector field may be written as the gradient of a scalar function. To determine if a vector field is a conservative vector field, measure the curl of the field. IF the curl is zero, the vector field is conservative! Consider the electric field, **E**, generated by a positive point particle in the xy-plane as shown in the figure below.

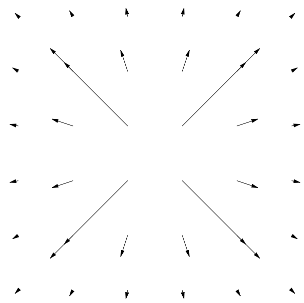


Figure 1: **E**-field due to positively charged particle.

From multi-variable calculus:

$$\nabla \times \mathbf{E} = 0 \tag{5}$$

$$\Rightarrow \mathbf{E} \text{ is a conservative vector field.} \tag{6}$$

$$\nabla \times (\nabla V) = 0. \tag{7}$$

$$\Rightarrow \mathbf{E} = -\nabla V \quad (V \text{ is the electric potential.}) \tag{8}$$

NOTE: The negative sign was determine physically by using the relationship between work and potential energy.

The electric field generated by charges is a conservative field; therefore, it possess zero "curl" but nonzero "divergence". The electric field in Fig. 1 clearly has a positive divergence at the point where the charge is located and a zero curl since the electric field lines do not "rotate" around the charge.

The electric field in Eq. 2 is not a conservative field since induced electric fields do not begin or end at a point in space (i.e., no charge produced the electric field).

The magnetic field in Eq. 4 is not a conservative field since the curl of the magnetic field is not zero. We can see this better if we consider the magnetic field of a very long, straight wire carrying current.

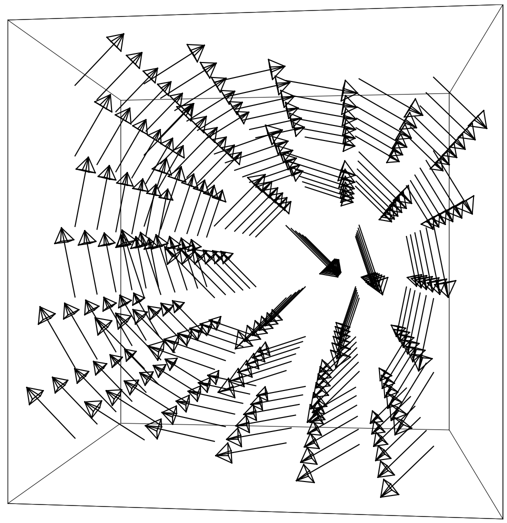


Figure 2: **B**-field due to current carrying wire.

Clearly the magnetic field is curling around the wire. Note: The wire is not drawn in the figure, but current through the wire is flowing into the page.

$$\nabla \times \mathbf{B} \neq 0.$$

Since no magnetic monopoles exist, then we can state that the divergence of the magnetic field is always zero. (There exists NO such figure like Fig. 1 for the magnetic field.)

$$\nabla \cdot \mathbf{B} = 0.$$

Maxwell's Equations in Differential Form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad (\rho \text{ is the charge density}). \tag{9}$$

$$\nabla \times \mathbf{E} = -\frac{d\,\mathbf{B}}{dt} \tag{10}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{11}$$

$$\nabla \times \mathbf{B} = \mu_0\, \jmath + \mu_0\, \varepsilon_0\, \frac{d\,\mathbf{E}}{dt} \qquad (\jmath \text{ is the current density}) \tag{12}$$

To understand how Eqs. 1- 4 are related to Eqs. 9- 12, we need to understand two major theorems of multi-variable calculus. These are: Gauss's theorem (Fundamental Theorem for Divergences) and Stoke's theorem (Fundamental Theorem for Curls).

Gauss's Theorem

$$\boxed{\int_{Volume} (\nabla \cdot \mathbf{F})\, dV = \oint_{Surface} \mathbf{F} \cdot d\mathbf{A}} \tag{13}$$

Geometrical Interpretation: If **F** represents the flow of an incompressible fluid, then the **flux** of **F** (the right hand side of Eq. 13) is the total amount of fluid passing out through the surface, per unit time. The divergence measures the "spreading out" of the vectors from a point. If there are many faucets within a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region. In essence, the divergence theorem states:

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface}) \tag{14}$$

Stokes's Theorem

$$\boxed{\int_{Surface} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_{\substack{boundary\\ \text{line}}} \mathbf{F} \cdot d\mathbf{l}} \tag{15}$$

Geometrical Interpretation: The curl measures the rotation of the vectors **F**. The integral of the curl over some surface (the *flux* of the curl *through* the surface) is a way of representing the TOTAL AMOUNT OF ROTATION. We can determine the total amount of rotation by going around the **edge** and determining how much the flow is following the boundary. The figure below may help with this mnemonic.

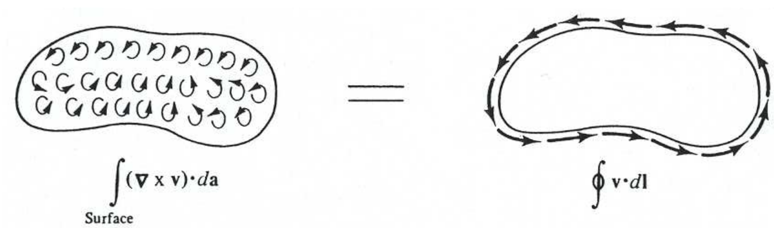


Figure 3: Stoke's Theorem Mnemonic

Applying the Fundamental Theorem for Divergences to Eq. 9, we obtain the following:

$$\int_{Volume} (\nabla \cdot \mathbf{E})\, dV = \int_{Volume} (\frac{\rho}{\varepsilon_0}) dV \equiv \frac{Q_{\text{enclosed}}}{\varepsilon_0} \tag{16}$$

$$\int_{Volume} (\nabla \cdot \mathbf{E})\, dV \overset{\substack{\text{Divergence}\\ \text{Theorem}}}{=} \oint_{Surface} \mathbf{E} \cdot d\mathbf{A} \tag{17}$$

$$\Rightarrow \oint_{Surface} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \tag{18}$$

We have reproduced Eq. 1.

Applying the Fundamental Theorem for Curls to Eq. 12, we obtain the following:

$$\int_{Surface} (\nabla \times \mathbf{B}) \cdot d\mathbf{A} = \int_{Surface} (\mu_0 \jmath + \mu_0 \varepsilon_0 \frac{d\mathbf{E}}{dt}) \cdot d\mathbf{A} \tag{19}$$

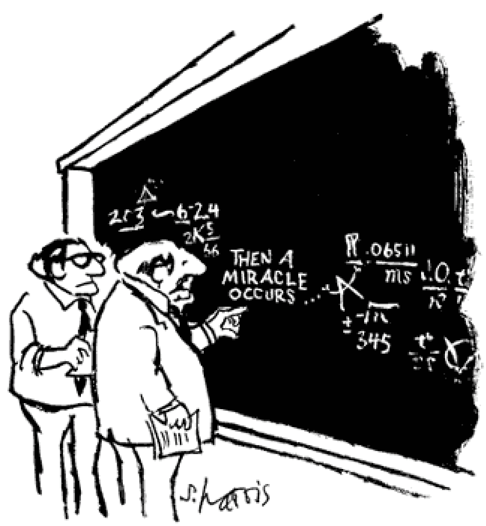
$$\text{but } \int_{Surface} (\mu_0 \jmath + \mu_0 \varepsilon_0 \frac{d\mathbf{E}}{dt}) \cdot d\mathbf{A} = \mu_0 \int_{Surface} \jmath \cdot d\mathbf{A} + \mu_0 \varepsilon_0 \int_{Surface} \frac{d\mathbf{E}}{dt} \cdot d\mathbf{A}$$

$$= \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_{Surface} \mathbf{E} \cdot d\mathbf{A} \tag{20}$$

$$\int_{Surface} (\nabla \times \mathbf{B}) \cdot d\mathbf{A} \overset{\substack{\text{Stoke's} \\ \text{Theorem}}}{=} \oint_{\substack{\text{boundary} \\ \text{line}}} \mathbf{B} \cdot d\mathbf{l}$$

$$\Rightarrow \oint_{\substack{\text{boundary} \\ \text{line}}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_{Surface} \mathbf{E} \cdot d\mathbf{A} \tag{21}$$

We have reproduced Eq. 4.



"I think you should be more explicit here in step two."

Figure 4: I hope this helps you out. Enjoy your summer.