Physics 102
Exam 1, February 22, 2000
Time allowed: 90 minutes

1. Print your LAST and FIRST names on the front of your blue book and on the multiple choice question sheet.
2. Answer the two free-response questions in the blue book. Answer the multiple choice questions by circling the single most nearly correct answer on the question sheet.
3. Consult no books or notes of any kind. You may use a hand calculator in non-programmed and non-graphing mode.
4. Do NOT take any of the test materials out of the room at any time.
5. Write and sign the pledge on the front of your blue book.
6. Show your work on the free-response questions. Even correct answers without explanation may be denied credit. There will be no partial credit on the multiple choice questions.

35. A sphere of radius \( R \) is centered at the origin. It carries a spherically symmetric, positive, volume charge density \( \rho(r) = \rho_0 r \) where \( \rho_0 \) is a constant, and \( r \) is the distance from the origin.

(a) Determine the total charge \( Q \) on the sphere.
(b) Determine the electric field \( \vec{E} \) everywhere in space.
(c) Sketch the electric field \( \vec{E} \) as a function of \( r \).

Now suppose that, in addition to the sphere, an infinite line of charge, carrying negative linear charge density \( -\lambda \), runs parallel to the x-axis a distance \( R/2 \) above the axis, as shown below. The presence of the line charge does not affect the charge distribution in the sphere.

(d) Determine the total electric field \( \vec{E} \) at the point \( P_1 \), which is outside the sphere, at \( x = a, y = a \), with \( a > R \).
(e) Determine the total electric field \( \vec{E} \) at the point \( P_2 \), inside the sphere, at \( x = 0, y = -R/2 \).
\[ P = P_0 n \]

(a) \[ Q = \int p dv \quad dv = 4\pi n^2 dr \]

\[ Q = P_0 \int (4\pi n^2 n) dr \]

\[ Q = 4\pi P_0 \int n^3 dr = 4\pi P_0 R^4 \]

\[ Q = P_0 \pi R^4 \]

(b) Apply Gauss's Law both inside and outside the sphere.

Inside:
\[ \int (E \cdot n) dA = 4\pi n^2 E = \frac{Q_{\text{enc}}}{\varepsilon_0} = \int \frac{p dv}{\varepsilon_0} \]

\[ 4\pi n^2 E = \frac{P_0}{\varepsilon_0} \int n (4\pi n^2) dr = 4\pi P_0 \frac{R^4}{n} \]

\[ 4\pi n^2 E_n = \frac{P_0}{\varepsilon_0} \frac{R^4}{n} \]

\[ E_n = \frac{P_0 n^2}{4 \varepsilon_0} \]

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\[ \overrightarrow{E} = \frac{P_0 n^2 \hat{r}}{4 \varepsilon_0} \quad n < R \]
Outside ($\lambda > R$), all the charge is enclosed

\[ \oint \overrightarrow{E} \cdot d\overrightarrow{A} = \frac{Q}{\varepsilon_0} = \frac{Q}{\varepsilon_0} = \frac{P_0 R^4}{\varepsilon_0 R^2} \]

\[ \overrightarrow{E} = \frac{Q}{4\pi \varepsilon_0 R^2} \]

\[ \lambda > R \]

Inside, $|E|$ increases quadratically; outside it falls as $\frac{1}{R^2}$.

(d)

\[ P_1 (x=a, y=a) \]

\[ a = \frac{R}{2} \]
At \( P_1 \), there will be contributions to the field due to both the sphere and the line of charge.

\[ E \text{ due to the line:} \]

\[ \Phi_E = \oint (E \cdot d\mathbf{A}) dA = \frac{Q_{	ext{enclosed}}}{\varepsilon_0} = \frac{-\lambda l}{\varepsilon_0} \]

\[ \nabla \cdot E = \frac{-\lambda l}{\varepsilon_0} \]

\[ E = \frac{-\lambda}{4\pi \varepsilon_0 r} \]

The (-) sign means the field is radially inward toward the wire.

At \( P_1 \), \( n = \hat{a} - \frac{\hat{r}}{r} \), and the direction will be the negative \(-y\) direction.

\[ E \text{ (due to line)} = \frac{-\lambda}{4\pi \varepsilon_0 (\hat{a} - \hat{r}/r)} \hat{y} \]
(d) Can't

Now we need to find $E$ at $P_2$ due to the sphere.

Referring to (a)

$E_{(due\ to\ sphere)} = \frac{\rho_0 R^4}{4 \varepsilon_0 R^2} \hat{\hat{r}}$ where $R^2 = 2a^2$ at $P_2$.

$E_y_{(due\ to\ sphere)} = \frac{\rho_0 R^4}{2 \varepsilon_0 a^3} \cos \theta \quad E_y_{(sphere)} = \frac{\rho_0 R^4}{2 \varepsilon_0 a^3} \sin \theta$

At $P_2$, $\theta = 45^\circ$ so we get

\[
\begin{align*}
E_x_{(sphere)} &= \frac{\rho_0 R^4}{8 \sqrt{2} \varepsilon_0 a^3} \hat{\hat{r}} \\
E_y_{(sphere)} &= \frac{\rho_0 R^4}{8 \sqrt{2} \varepsilon_0 a^3} \hat{\hat{j}}
\end{align*}
\]

The total field is the vector sum:

\[
E_{total} = \frac{\rho_0 R^4}{8 \sqrt{2} \varepsilon_0 a^3} \hat{\hat{r}} + \left[ \frac{\rho_0 R^4}{8 \sqrt{2} \varepsilon_0 a^3} - \frac{\lambda}{2 \pi \varepsilon_0 (a - R)} \right] \hat{\hat{j}}
\]

(e) At $P_2$, there will be contributions from both the sphere and the line of charge.

From (d) we know that the field due to the line at $P_2$ is

\[
E_{(line)} = \frac{\lambda}{2 \pi \varepsilon_0 R} \hat{\hat{j}} \quad \text{direction is +y}
\]
(c) Cont'd

From (a) we know that $E$ due to the sphere for $r < R$ is

$$
E^{\text{sphere}}(r) = \frac{\rho_0 n^2}{4 \pi \varepsilon_0} \hat{r}
$$

$r = \frac{R}{2}$ at $P_2$

At $P_2$

$$
E^{\text{sphere}}(P_2) = \frac{\rho_0 R^2}{16 \varepsilon_0} \hat{-y}
$$

direction is $-\hat{y}$

Then the total field at $P_2$ is

$$
E_{\text{tot}} = \left( \frac{\lambda}{2 \pi \varepsilon_0 R} - \frac{\rho_0 R^2}{16 \varepsilon_0} \right) \hat{r}
$$

at $P_2$
II. A positive point charge of magnitude $q$ is located on the positive y-axis a distance $a$ from the origin. Take the zero of potential to be at infinity.

$f$ (a) Determine an expression for the potential $V(x, y)$ for an arbitrary point $(x, y)$ in the x-y plane.

$g$ (b) Sketch $V(x)$ for all points on the x-axis.

$h$ (c) Determine the electric field $E_x$ for an arbitrary point on the x-axis.

$i$ (d) Sketch the x-component of $E$ as a function of $x$ for all points on the x-axis.

Now suppose we bring in two more identical charges and place them on the x-axis at $x = -a$ and $x = a$, as shown below.

$j$ (e) How much work must be done to assemble this three-charge configuration?

$k$ (f) Determine an expression for the potential $V(x, y)$ for an arbitrary point $(x, y)$ in the x-y plane for this new configuration.

$l$ (g) The charge at $x = a, y = 0$ is released and moves off to infinity. What is its kinetic energy when it is very far away from the other charges?
II. 

(a) \[ V(x, y) = \frac{\frac{2 \kappa}{q_i}}{\sqrt{x^2 + (y-a)^2}} \]

(b)  

(c) \[ E_x = \frac{\kappa q x}{(x^2 + a^2)^2} \quad E_y = \frac{-\kappa q a}{(x^2 + a^2)^{3/2}} \]

\( E \) at \( P \) will have both \( x \) and \( y \) components.
For \( x < 0 \), \( E_x \) changes sign but \( E_y \) does not.
The above expressions have the right sign for all values of \( x \).

\[
\frac{E_x(x)}{E_y} = \frac{h \gamma q x}{(x^2 + a^2)^{3/2}} \hat{x} - \frac{h \gamma q a}{(x^2 + a^2)^{3/2}} \hat{y}
\]

\( E_y \to 0 \) at \( x = 0 \) and at \( x = \pm a \)
\( E_x > 0 \) for \( x > 0 \)
\( E_x < 0 \) for \( x < 0 \)

Work to assemble = potential energy of system

\[
U = \frac{h \gamma^2 q^2}{2a} + \frac{3h \gamma^2 q^2}{\sqrt{2}a} = \frac{h \gamma^2 q^2}{a} \left[ \frac{1}{\sqrt{2}} + \sqrt{2} \right]
\]

\[
U = \frac{h \gamma^2 q^2}{a} \left[ \frac{1}{\sqrt{2}} + \sqrt{2} \right]
\]
\[ V(x,y) = \frac{\hbar q_y}{\sqrt{x^2 + (y-a)^2}} + \frac{\hbar q_y}{\sqrt{(x-a)^2 + y^2}} + \frac{\hbar q_y}{\sqrt{(x+q_1)^2 + y^2}} \]

Apply energy conservation.

Initial energy \( E_i = U_i = \frac{\hbar q_y^2}{2a} + \frac{2\hbar q_y^2}{\sqrt{a^2}} \)

Final energy \( E_f = U_f + K \)

The final potential energy is

\[ U_f = \frac{\hbar q_y^2}{\sqrt{a^2}} \]

\[ E_i = E_f \]

\[ \frac{\hbar q_y^2}{2a} + \frac{2\hbar q_y^2}{\sqrt{a^2}} = \frac{\hbar q_y^2}{\sqrt{a^2}} + K \]

\[ K = \frac{\hbar q_y}{2a} + \frac{\hbar q_y}{\sqrt{a^2}} \]

\[ K = \frac{\hbar q_y^2}{2a} \left[ \frac{1}{a} + \frac{1}{\sqrt{a^2}} \right] \]
TEN MULTIPLE CHOICE QUESTIONS. 3 points each. For each of the following questions, circle (on these sheets) the option that is most nearly correct.

(Questions 1-3) Consider the following configurations of charges. In each case, the charges are located at the vertices of an equilateral triangle and Point P is equidistant from the charges.

1. The magnitude $|\vec{E}|$ of the electric field at P is the least for the configuration

   a) A.  \[ E = 0 \]
   b) B.
   c) C.
   d) D.
   e) E.

2. Setting the potential $V = 0$ at distances very far from P, the potential at P has the greatest magnitude (absolute value of V) for the configuration

   a) A.  \[ \frac{q}{4\pi\varepsilon_0} \]
   b) B.
   c) C.
   d) D.
   e) E.

3. The electric field vector $\vec{E}$ at P points at the midpoint of two of the charges for the configuration

   a) A.
   b) B.
   c) C.
   d) D.
   e) E.
4. A charge \( q \) is moved a distance \( d \) between two points \( A \) and \( B \) by an external agent. In the process the external agent does work \( W \). We may therefore deduce that the potential difference between points \( A \) and \( B \) is

a) \( \frac{W}{q} \)
b) \( \frac{(Wq)}{d} \)
c) \( W\frac{d}{q} \)
d) \( \frac{q(d)}{W} \)
e) \( kq/d \)

5. A non-conducting hollow sphere of radius \( R \) carries a large charge \( +Q \), which is uniformly distributed on its surface. There is a small hole in the sphere. A small charge \( +q \) is initially located at a distance \( r \) from the center of the sphere. The work that must be done by an external agent to move the charge from its original position through the hole to the center of the sphere is

a) Zero
b) \( \frac{kqQ}{r} \)
c) \( \frac{kqQ}{R} \)
d) \( \frac{kq(Q-q)}{r} \)
e) \( \frac{kqQ}{R} - \frac{kqQ}{r} \)

6. A parallel-plate capacitor is attached to a battery of a given potential. With the battery still connected, the plates of the capacitor are pushed together until they are at half their original separation. As a result

a) The electric charge on the plates is doubled.
b) The electric charge on the plates is unchanged.
c) The electric charge on the plates is halved.
d) The energy stored in the capacitor is unchanged.
e) The potential difference between the plates is halved.

7. Some charge is deposited on a conductor. The charge then redistributes itself until it is at electrostatic equilibrium. Which of the following statements is now true?

a) The charge is evenly distributed over the surface of the conductor.
b) The strength of the electric field just outside the conductor is the same for every point on the surface of the conductor.
c) The electric potential inside the conductor is zero.
d) The electric field at the surface of the conductor is tangent to the surface.
e) The surface of the conductor is an equipotential surface.
(Questions 8-9) Three capacitors of capacitance $C_0$, $2C_0$, and $3C_0$ (as labeled) are arranged in a triangular network as shown. A battery of potential $V$ is attached across the terminals $a$ and $c$.

8. The equivalent capacitance between terminals $a$ and $c$ is
   a) $C_0/5$
   b) $2C_0/5$
   c) $C_0$
   d) $1C_0/5$
   e) $16C_0/5$

9. The electrostatic energy stored in the capacitor with capacitance $3C_0$ is
   a) $3C_0 V^2/5$
   b) $6C_0 V^2/5$
   c) $3C_0 V^2/25$
   d) $6C_0 V^2/25$
   e) $27C_0 V^2/50$

10. Consider a parallel-plate capacitor whose plate dimensions and spacing are unknown to you. Which following single bit of information will allow you to calculate the magnitude $|\vec{E}|$ of the electric field between the plates? Assume that you know all the fundamental constants of electrostatics.

   a) The electric flux between the plates.
   b) The total charge on either plate.
   c) The potential difference between the plates
   d) The surface charge density on either plate.
   e) The total energy stored in the capacitor.