1. A projectile of mass \( m \) moves to the right with speed \( v_i \) as indicated in the figure below. The projectile strikes and sticks to the end of a uniform, stationary rod of identical mass \( m \) and length \( d \) that is pivoted about a frictionless fixed axle through its center (see Fig(b) below). \((\text{The rotational inertia for a uniform rod of mass } M \text{ and length } L \text{ rotated about an axis perpendicular to the plane of the rod and through the center of the rod is } I = \frac{1}{12}M L^2)\)

(a) Find the magnitude of the angular velocity of the system immediately after the collision.

(b) Determine the fractional loss of mechanical energy due to the collision.

(c) Determine the impulse delivered to the rod by the pivot during the collision.

(a). Since there are no external torques acting on the system of mass and rod, we can consider the conservation of angular momentum for the system.

\[
\sum_0 \tau = \frac{d\vec{L}}{dt} = 0 \tag{1}
\]

\[
\Rightarrow \vec{L} = \text{constant} \tag{2}
\]

\[
\vec{L}_f = \vec{L}_i \tag{3}
\]

\[
\vec{L}_f = I_{\text{system,O}} \vec{\omega} \tag{4}
\]

\[
\vec{L}_f = -\left( \frac{1}{12} m_{\text{rod}} L^2 + m_{\text{ball}} \left( \frac{L}{2} \right)^2 \right) \omega \hat{k} \tag{5}
\]

\[
\vec{L}_f = -\frac{1}{3} m L^2 \omega \hat{k} \tag{6}
\]
The only object moving initially is the ball (it carries momentum). Since the ball is physically displaced from the pivot at the instant of impact, it carries angular momentum with respect to the pivot right before the collision.

\[ \vec{L}_i = \vec{r}_i \times \vec{p}_i \]  

(7)

\[ \vec{L}_i = m v \frac{L}{2} (\hat{j} \times \hat{i}) \]  

(8)

\[ \vec{L}_i = -m v \frac{L}{2} \hat{k} \]  

(9)

Equating Eq.9 with Eq.6 gives an expression for \( \omega \):

\[ \frac{1}{3} m L^2 \omega = m v \frac{L}{2} \]  

(10)

\[ \omega = \frac{3 v}{2 L} \checkmark \]  

(11)

(b). In order to calculate the fractional loss of mechanical energy due to the collision, we need to calculate the initial and final kinetic energy for the system.

\[ \Delta K = \text{Energy loss.} \]  

(12)

\[ K_i = \frac{1}{2} m v^2 \]  

(13)

\[ K_f = \frac{1}{2} I \omega^2 = \left( \frac{1}{2} \right) \left( \frac{1}{3} m L^2 \right) \left( \frac{3 v}{2 L} \right)^2 \]  

(14)

\[ K_f = \frac{3}{8} m v^2 \]  

(15)

\[ K_f - K_i = \Delta K = -\frac{1}{8} m v^2 \]  

(16)

\[ \frac{\Delta K}{K_i} = -\frac{1}{4} \text{ (or 25% loss.)} \checkmark \]  

(17)
(c). To determine the impulse delivered to the rod by the pivot during the collision, we need to go back to the definition of impulse (center-of-mass):

\[
\Delta \vec{p}_{\text{CofM}} = 2m (\vec{v}_{i,\text{CofM}} - \vec{v}_{i,\text{CofM}}) \tag{18}
\]

\[
\vec{v}_{i,\text{CofM}} = \omega \frac{L}{4} \hat{i} \tag{19}
\]

\[
\vec{v}_{i,\text{CofM}} = \frac{3}{8} v \hat{i} \tag{20}
\]

\[
2m \vec{v}_{i,\text{CofM}} = m v \hat{i} \tag{21}
\]

\[
\vec{v}_{i,\text{CofM}} = \frac{1}{2} v \hat{i} \tag{22}
\]

\[
\Delta \vec{p}_{\text{CofM}} = 2m v \left( \frac{3}{8} - \frac{1}{2} \right) \tag{23}
\]

\[
\Delta \vec{p}_{\text{CofM}} = -\frac{1}{4} m v \hat{\imath} \tag{24}
\]
2. A ladder 20 ft long weighing 80 lbs leans against a friction-free wall. Its foot is 12 ft from the base of the wall (see figure below) resting on a rough surface. The coefficient of static friction between the surface and the foot of the ladder is \( \mu_s = 0.4 \). Assume the ladder's mass is uniformly distributed. (Note: \( g = 32 \text{ft/sec}^2 \).)

(a) If a man of weight 160 lbs is standing at the midpoint of the ladder, is it in equilibrium?
(b) How far can the man climb up the ladder before it slips?

(a). Begin by drawing the FBD for the ladder.
Using the conventional right-handed-coordinate system, we need to apply Newton’s 2\textsuperscript{nd} law to the ladder and torques about a point.

\[ \sum_x F : f - N_W = 0 \quad \text{(in equilibrium.)} \quad (25) \]

\[ \sum_y F : N_f - (W_M + W_L) = 0 \quad (26) \]

\[ \sum_z \tau_O : -r_{\perp,W} (W_M + W_L) + r_{\perp,N} (N_W) = 0 \quad (27) \]

\[ r_{\perp,W} = 10 \cos \theta \quad (28) \]

\[ r_{\perp,N} = 20 \sin \theta \quad (29) \]

\[ \theta = \cos^{-1} \frac{12}{20} \approx 53.13^\circ \quad (30) \]

where \( r_{\perp,W} \) represents the perpendicular distance from the pivot point \( (O) \) to the point of application of the weight and \( r_{\perp,N} \) represents the perpendicular distance from the pivot point \( (O) \) to the point of application of the normal force exerted by the wall.

To check for equilibrium, one needs to verify that \( f \geq N_W \). Using Eq. 27 one can find an expression for \( N_W \).

\[ N_W = \frac{10 \cos \theta (W_M + W_L)}{20 \sin \theta} = 90 \text{lbs} \quad (31) \]

\[ f \geq N_W = \mu_s N_f \geq N_W \quad (32) \]

\[ \Rightarrow \mu_s \geq \frac{N_W}{N_f} = \frac{90}{240} \quad (33) \]

\[ \mu_s \geq 0.375 \quad (34) \]

Since \( \boxed{\mu_s = 0.4 \geq 0.375} \), the system of ladder plus man is in equilibrium.
(b). To find how high the man can move up the ladder, we will call his position along the ladder from the point \( O \) “\( x \)”. Rewriting the torque equation

\[
\sum_x \tau_O : r_{\perp,WM} (W_M) - r_{\perp,WL} (W_L) + r_{\perp,N} (N_W) = 0. \tag{35}
\]

\[
r_{\perp,WM} = 10 \cos \theta \tag{36}
\]

\[
r_{\perp,WL} = x \cos \theta \tag{37}
\]

\[
r_{\perp,N} = 20 \sin \theta \tag{38}
\]

\[
N_W = \mu_s (W_M + W_L) \quad \text{(Max. Frictional force.)} \tag{39}
\]

\[
W_M x \sin \theta = 20 \mu_s (W_M + W_L) \cos \theta - 10 W_L \sin \theta \tag{40}
\]

\[
\Rightarrow x = 11 \text{ft} \quad \checkmark \tag{41}
\]

3. A block suspended from a spring (see the figure below) is executing simple harmonic motion (SHM) with frequency \( f = 4 \) Hz (4 cycles/sec) and amplitude 0.07 m. A small rock of negligible mass is placed on the block at its lowest position.

(a) Determine the distance above the block’s equilibrium position at which the rock loosens contact with the block.

(b) Determine the speed of the rock when it breaks contact with the block.

(a) Begin by drawing the free-body-diagram for the small rock.

Summing all forces on the rock.
\[ \sum F_y : N - mg = ma_y \] (42)

Since the rock is connected to the block (i.e., \( N \neq 0 \)) \( a_y \) is the acceleration of the block. The block is executing SHM so

\[ a_y = \frac{d^2 y}{dt^2} = -\omega^2 y \] (43)

\[ y = A \cos (\omega t + \phi_0) \] (44)

When the rock just leaves the block means that \( N \to 0 \) so this means

\[ N^0 - mg = ma_y \] (45)

\[ \Rightarrow -g = a_y \] (46)

\[ \Rightarrow -g = -\omega^2 y \] (47)

\[ \Rightarrow y = \frac{g}{\omega^2} \] (48)

We were given the amplitude \((A)\) and frequency \((f)\) of the harmonic oscillator. Using \( \omega = 2\pi f = 8\pi \) yields:

\[ y = \frac{g}{64\pi^2} = 0.0156 \text{ m} \] (49)

(b) To find the speed at this instant (i.e., \( y = 0.015 \text{ m} \)), we need to determine the speed as a function of \( y \).

\[ v = \frac{dy}{dt} = -\omega A \sin (\omega t + \phi_0) \] (50)

\[ v^2 = \omega^2 A^2 \sin^2 (\omega t + \phi_0) = \omega^2 A^2 (1 - \cos^2 (\omega t + \phi_0)) \] (51)

\[ v^2 = \omega^2 A^2 - \omega^2 A^2 \cos^2 (\omega t + \phi_0) = \omega^2 A^2 - \omega^2 y^2 \] (52)

\[ v = \omega \sqrt{A^2 - y^2} \] (53)

\[ \Rightarrow v = 8\pi \sqrt{(0.07 \text{ m})^2 - (0.0156 \text{ m})^2} \] (54)

\[ \Rightarrow v = 1.72 \text{ m/s} \] (55)