III. Uncertainty and Significant Figures

> "Errors using inadequate data are much less than those using no data at all."
>
> C. Babbage

The results of any physical measurement process will be inaccurate to a greater or lesser degree. We will frequently want to know if two measured quantities are the same or different, or if a set of data is adequately described by some mathematical model. Because our measured numbers cannot be exact, this comparison is not as simple as asking if two or more numbers are equal. Rather, one must investigate the uncertainties in the measured quantities and then ask if the deviations are within the range of those uncertainties. If the deviations are smaller than the uncertainties we say that there is agreement, otherwise there is a significant difference. This section will introduce you to the rudiments of that comparison process for some simple situations. Later in your career you will learn more sophisticated statistical treatments for more difficult problems.

ESTIMATING UNCERTAINTIES

There are many possible sources of error and uncertainty in measurement, but they can be roughly classified into three groups. The most obvious is the mistake. Measuring the wrong quantity, writing down the wrong number or misreading a dial all lead to wrong answers. The cure is to be careful, and to check that all results are reasonable. Systematic errors form another group. Here the apparatus or operating procedure gives a result which always differs from the correct value in a predictable way. For example, a wood meter stick might have expanded due to high humidity, causing lengths to be read as shorter than they are. This type of error is usually discovered by comparing different measurement methods, or sometimes by a detailed study of the instruments and procedures. Even without mistakes or systematic errors, one finds that measurements are not exactly repeatable. This residual uncertainty is essentially random, so it is often called "random error" even though it is inherent in the measuring process and not a mistake of some sort. The rest of this section describes how to estimate and deal with this type of uncertainty.

The idea of repeating a measurement several times to check for variation is probably familiar. The arithmetic mean of the measured values is usually taken as the best estimate of the actual value, and the range or standard deviation is an estimate of uncertainty. For example, we might wish to know the weight of a rodent for a biological study. If we put the animal on a scale with a digital indicator, the readings will fluctuate as it wiggles. Reading the figures several times and averaging should then give us a better estimate of the weight than a single reading. We could also use the range of readings we get to estimate the uncertainty in our average value. In other situations we might have to repeat an entire measurement process to get several values to average, but the principle is the same.

The resolution of the measuring instrument may also limit the precision of a measurement. For example, if we use a digital scale to weigh an inanimate object the reading will probably be very steady, and we will probably get very nearly the same reading if we reweigh the object later. Even so, we cannot determine the weight more closely than the value of the lowest digit on the indicator, whether or not we average many readings. With an analog instrument, like a meter stick, it might be possible to interpolate between the smallest scale divisions, but that will not give infinite precision either. In these cases we can take the uncertainty to be the smallest increment that can be read from the scale.

SIGNIFICANT FIGURES

Having made a measurement and estimated the uncertainty, we also need to know how to record and report the result. The usual procedure is to write down the value ± the uncertainty, using only the appropriate number of digits for each. For example, in weighing the rodent we might find an average for 5 readings of 74.54 gm, with a variation of 1.3 gm. From these numbers we deduce that the "true" weight lies somewhere in the range 73.89 gm to 75.19 gm. Given this range, it is clearly overstates our precision to give the weight as 74.54 gm. Rather, one should use 74.5 gm ± 1.3 gm, or even 74 gm ± 1 gm when reporting the result. The exact number of digits to keep is not rigidly fixed, but the basic principle is to give only significant digits, those that represent a value larger than the uncertainty in the quantity. Writing down more digits just because they are on the calculator display is both misleading and a waste of effort.

PROPAGATION OF UNCERTAINTIES

In many instances we will want to calculate other quantities from measured numbers. As you might expect, the uncertainties in the measurements result in uncertainties in those calculated values. This is sometimes called "error analysis", and can be done rigorously, but a simple approximation is more useful here. Roughly speaking, if a quantity can be measured only to 10%, then any calculation involving that quantity can give results known only to about 10% also. That is, your calculated results are never going to be better than the data that went into them.

Two approximate rules will cover most of the computations that arise in this lab. For multiplication and division, the relative error in the result is the sum of the relative errors in the inputs. For addition or subtraction, the absolute error in the result is the sum of the absolute errors in the inputs. (This is why it is usually bad when your answer is the difference of two large numbers.) These rules can be summarized in a set of formulae:

$$z = xy \quad \text{or} \quad z = x/y \qquad \frac{\Delta z}{z} = \frac{\Delta x}{x} + \frac{\Delta y}{y} \qquad \text{(III-1)}$$

$$z = x \pm y \qquad \Delta z = \Delta x + \Delta y \qquad \text{(III-2)}$$

where Δx, Δy are the uncertainties in the inputs and Δz is the uncertainty in the calculated quantity.

An example of this whole process may be useful. Suppose we wish to measure the average speed of a ball rolling down an inclined plane by measuring the travel time between two marks on the plane. Using a meter stick, we measure the distance between the marks to be 10.0 ± 0.1 cm. The ± 0.1 cm means the true distance could be anywhere in the range 9.9 cm to 10.1 cm. Given the width of the marks and the smallest divisions on the ruler we can't do any better. Next, using a stopwatch, we measure the time interval. The watch reads to 0.01 s, but in several trials we get values varying by 0.18 s, so we estimate the time interval as 1.5 ± 0.1 s. Note that the error estimates were arrived at by studying the apparatus in one case, and by observing variations in results in the other. We calculate the average speed to be 10.0 cm/1.5s = 6.7 cm/s, using the best estimates of distance and time. Computing the percent uncertainties we find that the distance is known to about ± 1%, while the time is known only to ± 6%. The calculated result must then be uncertain by about ± 7%. We would then claim the speed is known to be 6.7 ± 0.5 cm/s.

You will have many opportunities to estimate errors in the course of your work this semester. You should not become tangled in details, but simply apply some common sense along the lines indicated to estimate the reliability of your results. Since all measurements are inaccurate to some degree, an ability to realistically estimate one's state of knowledge is essential to the scientific enterprise.