Extreme Events in Stock Market Fundamental Factors

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Abstract

We estimate a joint multivariate jump-diffusion model using daily data for three fundamental stock market factors: market return, value, and momentum. We focus on the description of risk represented by the joint dynamics of factor volatilities and extreme events. With regard to extreme events, we find (i) evidence of co-jumps through both the volatility channel and increased probabilities of simultaneous return jumps and, (ii) unlike the momentum factor, the value factor does not jump independent of the market return. We discuss implications of our empirical findings for factor investing and for possible sources of premia in the momentum and value factors. We find that the contribution of the extreme event risk to the value premium is negligible, while a significant portion of the momentum premium can be attributed to the extreme event risk.

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Introduction

Market return is universally regarded as the primary driver of systematic risk in asset prices. In light of this particular role, it is not surprising that the dynamics of market return, as approximated by returns on broad stock market indices, have been extensively studied in the literature. Various characteristics of daily market return dynamics have important asset pricing and risk management implications. First, market returns are largely unpredictable (see Kendall, 1953, and Fama, 1965, 1970, 1991). Second, the volatility of market return is not constant (see reviews by Bollerslev, Chou, and Krone, 1992, and Andersen, Bollerslev, and Diebold, 2003). Third, shocks to return volatility are negatively correlated with shocks to returns – the phenomenon known as the leverage effect (see Black, 1976, and Christie, 1982). Fourth, the presence of infrequent, large negative returns can be conveniently captured by models with jumps, e.g., Jarrow and Rosenfeld (1984), Jorion (1989), Andersen, Benzoni, and Lund (2002), and Chernov et al. (2003). Under the capital asset pricing model, these facts together allow us to quantify systematic risk in stock portfolios based on their market betas.

However, the voluminous literature on cross-sectional asset pricing suggests that there are likely to be other sources of priced risk that are not spanned by market return alone. Indirect evidence of this fact is the success of Fama and French’s (1993) factors in capturing a cross-section of expected stock returns. Given that systematic risk has a multi-factor structure, the next important step in characterizing systematic equity risk is characterizing the factors’ joint dynamics. This study considers two of the standard Fama-French factors: value and market return $^1$, plus the momentum factor of Carhart (1997). These factors show robust performance across markets and asset classes (e.g., Asness, Moskowitz, and Pedersen, 2013) and are useful in practice with the investment style known as factor investing (see Bender et al., 2013, Pappas and Dickson, 2015, Ang, 2013).

The literature on the joint risk dynamics of equity factors is not nearly as developed as the literature on market return dynamics. One notable exception is the recent paper by Christoffersen and Langlois (2013) who suggest a dynamic copula model with the GARCH specification of

$^1$We exclude size because of its lessened role in the considered sample from 1980 to 2013.
Bollerslev (1986) for volatilities and DCC (dynamic conditional correlation) specification of Engle (2002) for correlations. With their model, Christoffersen and Langlois (2013) study the time variation and asymmetry in factor correlations. Other related papers include Chan, Karcseki, and Lakonishok (1999), who compare forecasting models for factor covariances; and Ang and Chen (2002), who estimate upside and downside correlations with a market index of size and value-sorted portfolios.

This paper extends an important class of non-linear models, known as jump diffusions, from the univariate case of market return to the joint dynamics of multiple factors. Jump-diffusion models allow for a decomposition of innovations to returns into a continuous (diffusion) part and jump part. In turn, volatility innovations can also be decomposed into continuous and jump parts. Modeling the fundamental factors jointly provides a characterization of joint dynamics in both the diffusion components (i.e., standard fluctuations) and jump components (i.e., tail events). Additionally, jump-diffusion models assume a stochastic specification for volatility, which permits the study of contemporaneous correlations between the factors’ volatilities and returns.

This study focuses on co-interactions between tail events in factor returns. With our model, we are able to differentiate between co-jumps and individual jumps. We find statistical evidence of two types of co-jumps. First, a jump in the market return increases the probability of jumps in the other factors. The estimated probability of these co-jumps conditional on a market jump is close to one. Second, volatility jumps are correlated among factors. Such volatility co-jumps are characterized by coincident timing and correlated magnitudes across factors. Additionally, we found evidence for individual jumps in the momentum series. This evidence points to additional sources of risk in the momentum strategy.

Our empirical results are applied to establish the properties of factor portfolios during extreme events. That is, we consider diversification away from the market return and portfolios that are tilted to include the other two factors. The analysis of these portfolios during extreme events is impeded by the rarity of such events. However, our estimation results allow us to calculate a variety of summary statistics for returns on jump days. We conclude that adding the value factor to a market portfolio improves its performance during extreme events in terms of the average
loss, volatility, and value-at-risk (VaR). Similarly, adding the momentum factor reduces volatility and VaR. However, the momentum strategy has historically led to larger average extreme losses. Therefore, we conclude that only the momentum factor can be associated with some additional tail risk, while the value premium cannot. Furthermore, we document the jump leverage effect in momentum returns, that is, a negative correlation between extreme volatility spikes and concurrent extreme returns. Thus, the momentum returns exhibit additional tail risk and adjust for this type of risk.

Finally, the paper analyzes the extreme event premia in factors. Thus far, this question was most successfully addressed for market returns using information from derivative markets, as in Liu, Pan, and Wang (2005), Bollerslev and Todorov (2011), Seo and Wachter (2016), and Xu (2016). Because derivatives for momentum and value factors do not exist, this method cannot be extended beyond the market return. Therein we adopt the following approach. Information about joint dynamics of the market, momentum, and value returns is combined with the one-factor jump risk model to arrive at the magnitudes of the jump risk premia in momentum and value factors as functions of the market jump risk. We show that while the value premium cannot be attributed to jump risk, the momentum jump premium is positive and can be approximately 40%-80% of the market jump premium.

The estimation methods in this paper are most closely related to those in Eraker, Johannes, and Polson (2003). Eraker, Johannes, and Polson (2003) provide a blueprint for Markov Chain Monte Carlo (MCMC) estimation of univariate jump-diffusion models, as applied to the market return series. We extend this method to the multivariate case and provide a detailed description of the algorithm. Other papers that estimate multivariate jump-diffusion models for equity data include Johannes and Polson (2009) and Ait-Sahalia et al. (2015). Johannes and Polson (2009) estimate the multivariate Merton model with constant volatility. In contrast, our model includes time variation in volatility. Ait-Sahalia et al. (2014) assumes only one volatility component but allow for an extra feature in the dynamics of jumps, referred to as self-excitation. Our model includes simultaneous cross-excitation of jumps and allows for a more general volatility dynamic with a separate volatility component corresponding to each factor. Aguilar and West (2000) consider a rich factor model for exchange rate volatilities but limit their results to conditionally
Gaussian dynamics. The Euler discretization of our model nests the conditionally Gaussian case if jumps do not occur.

The paper proceeds as follows. Section 1 describes the factor series. Section 2 summarizes the properties of these data by fitting univariate jump-diffusion models. Section 3 introduces the multivariate model. Section 4 reports the estimation results. Section 5 discusses the implications of these results for factor investing. Section 6 evaluates the jump-risk premia in momentum and value factors. Section 7 provides conclusions.

1 Data

We consider stock market factors over the sample time period of January 1, 1980 through March 31, 2013 using daily excess returns for a total of 8,385 observations. We extend the period studied by Eraker, Johannes, and Polson (2003) to include the most recent decade as well as the years from 1980 to 1990. Therefore, our sample includes three extremely turbulent periods: 1987-1988, 1997-2002, and 2007-2009. The data were obtained from Kenneth French’s data library\(^2\). The market excess return (MKT) is the value weighted return of the common shares of all firms incorporated in the US in the CRSP database that are listed on the NYSE, AMEX or NASDAQ less the daily return from holding a one-month Treasury bill. The value factor (HML) is constructed as the returns to a zero net cost portfolio that takes a long position in two value portfolios and a short position in two growth portfolios. The momentum factor (UMD) is constructed as the returns to a zero net cost portfolio that takes a long position in two portfolios of winners over the past two months to one year and a short position in two portfolios of losers over the same time period.\(^3\)

Figure 1 plots the factor series and Table 1 reports summary statistics. Apparent similarities in the dynamics of the returns include small averages that are virtually zero in comparison to volatilities; co-movements in long-run volatilities, with the most recent increase starting around September 2008; and high kurtosis and negative skewness in the market and momentum re-

\(^2\)We thank Kenneth French for making these data available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

\(^3\)For more information on factor construction the reader is referred to the notes in Kenneth French’s data library.
turns. To further discern the properties of the factor returns, we proceed to the jump-diffusion decomposition of the three time series.

2 Univariate Properties of Factor Series

The stock market factors in our dataset are constructed as weighted averages of the same set of returns and, therefore, are expected to exhibit certain similarities in dynamics. It is reasonable for our analysis of their univariate properties to select a model known to fit the dynamics of one of these factors – the market return.

Stock market returns, as approximated by stock market indices, have been the subject of many econometric studies. Chernov et al. (2003) compare a comprehensive list of models and find that, within the affine family of models, the jump-diffusion model is best supported by the data. Although jumps are formally defined as discontinuities in the price path, the inclusion of jumps in the daily return model can be thought of as a way of incorporating large price movements. Within this framework we can explain the high kurtosis and negative skewness of the market portfolio by the existence of infrequent, large negative fluctuations.

Markov chain Monte Carlo (MCMC) estimation, as suggested by Eraker, Johannes, and Polson (2003), provides a convenient method for estimating jump-diffusion models. Among the models they consider (Heston, 1993, Bates, 1996, Duffie, Pan, and Singleton, 2000), they find that a model with contemporaneous jumps in returns and volatility (Stochastic Volatility Conditional Jump, SVCJ) is consistent with the observed data.

We apply the SVCJ model \(^4\) to all of the factor series \(F_t\) \(^5\) and corresponding spot variances \(V_t\):

\[
\frac{dF_t}{F_{t-}} = \mu dt + \sqrt{V_t} dB_t + \xi_f dN_t, \tag{1}
\]

\[
dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dB^v_t + \xi_v dN_t,
\]

\(^4\)In Eraker, Johannes, and Polson (2003), the model is applied to logarithmic returns. The factor returns in our dataset are computed as differences between the long and short positions of the factor portfolios. Therefore, taking the logarithm is neither well-defined (if the loss is larger than the value of the position) nor it can be interpreted as the difference in logarithmic returns. Herein, the SVCJ model is applied to simple returns. For notational brevity, we continue to use a symbol \(dF_t/F_t\) to denote the difference between instantaneous returns to the long and short positions.

\(^5\)With brevity, we omit factor-specific subscripts.
where $V_t = \lim_{s \uparrow t} V_s$ is the left limit of spot volatility, $B_t$ and $B_v^t$ are standard Brownian motions correlated with coefficient $\rho$, $N_t$ is a Poisson process with intensity $\lambda$, and $\xi^f_t$ and $\xi^v_t$ are the jump sizes of returns and variance. The diffusion components consist of the shocks $\sqrt{V_t} \cdot dB_t$ and $\sigma_v \sqrt{V_t} \cdot dB_v^t$. The jump components of the model are given by $\xi^f_t \cdot dN_t$ and $\xi^v_t \cdot dN_t$. The parameters $\mu$, $\kappa$, $\theta$, and $\sigma_v$ represent the average instantaneous return in the absence of jumps, mean reversion of variance, the normal level of variance, and the volatility of variance, respectively. The processes $\xi^f_t$ and $\xi^v_t$ are the jump sizes. To ensure the positivity of $V_t$, $\xi^v_t$ is modeled as an exponential random variable $\xi^v_t \sim \exp(\mu_v^J)$ with the density $f_{\xi^v_t}(x) = \mu_v^J \exp(-\mu_v^J x)$ for $x \geq 0$. The sizes of jumps in prices are allowed to be correlated with the size of jumps in volatility, i.e., $\xi^f_t$ is modeled as conditionally normal $\xi^f_t | \xi^v_t \sim N(\mu^J + \rho^J \xi^v_t, (\sigma^J)^2)$, where $\xi^v_t$ is directly incorporated into the mean return jump size.

The model above allows for a decomposition of return variance into two components: one related to diffusion terms and one related to jumps. The conditional variance of returns is

$$\frac{1}{dt} V_t \left( \frac{dF_t}{F_t} \right) = V_t + \lambda \mathbb{E} \left( \xi^f_t \right)^2 = V_t + \lambda \left( (\mu^J)^2 + 2 \left( \frac{\rho^J}{\mu_v^J} \right)^2 + 2 \frac{\mu^J \rho^J}{\mu_v^J} + (\sigma^J)^2 \right).$$

Replacing $V_t$ by its expectation $\theta + \frac{\lambda}{\kappa \mu_v^J}$, we find that the contribution of the diffusion component to the variance is $\theta$ and the contribution of the jump component is $\lambda \left( \frac{1}{\kappa \mu_v^J} + \mathbb{E} \left( \xi^f_t \right)^2 \right)$.

It should be noted that, apart from jumps in returns, the SVCJ model also accommodates other types of extreme return dynamics. First, shocks $dB_t$ can be negatively related to volatility shocks due to the leverage effect $\rho < 0$. That is, prices adjust downwards in response to higher volatility. Additionally, an increase in the level of $V_t$ naturally increases the occurrence of extreme positive and negative returns.

To estimate the model, we use the Euler discretization scheme on daily data in percentage form and adopt the same estimation steps as in Eraker, Johannes, and Polson (2003). Table 2 reports parameter estimates and Table 3 provides a variance decomposition of daily returns into diffusion and jump components. Figure 2 plots the estimated latent variance series together with the estimated volatility jump markings, and Figure 3 plots the estimated probabilities of

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6The reported variance series correspond to $\mathbb{E}(V_t \cdot \frac{F_i - F_{i-1}}{F_{i-1}}, i = 1, .., T)$, i.e., estimates of $V_t$ based on observed returns.
Market Return. We begin by discussing results for the market portfolio which serves as a reference point for the other factors considered herein. The implied daily return for the market index, given the estimated parameters, is approximately 0.023% daily (5.8% annualized). Similar to prior studies, we find evidence of a strong leverage effect between returns and variance shocks, with a negative correlation coefficient of $\rho = -0.619$. The normal level of variance is $E(V_t) = 1.09$, which corresponds to a daily value-at-risk (VaR) of around -1.71%. The decomposition in Table 3 complements previous findings that the market portfolio is subject to sporadic, large changes in value that are generally negative. Given the estimate for the parameter $\lambda$, we should expect to see approximately 66 jumps in our sample. Jumps also constitute a large portion (45%) of the total market variance.

For the market return, we have a total of 12 days within the sample in which the probability of a jump is over 80%. They include the stock market crash of 1987 (October 19), the mini-crash of 1989 (October 13), and the Asian Crisis of 1997 (October 27); as well as dates in 2000 (January 4) and 2007 (February 27, Chinese correction) marking the first crack in the dot-com bubble and the start of the global financial crisis, respectively. Less apparent crash days with jump probabilities of over 80% include the market correction on July 7, 1986, “Black Thursday” (September 11, 1986) – subsequently attributed to news about rising inflation, the market liquidity break (Christie and Schultz, 1998) on November 15, 1991, and the market reaction to news about the federal funds rate (Pakko and Wheelock, 1996) on February 4, 1994.

Our estimates of market variance are plotted in the top panel of Figure 2. We find increases in variance in the late 1990s through the early 2000s, and at the onset of the financial crisis in 2008. On top of $V_t$ dynamics, Figure 2 marks days on which the estimated probability of a jump exceeds 20% and the estimated value of $\xi_v$ is not zero. As follows from the figure, high-volatility periods begin with jumps in volatility.

To further clarify the interpretation of jump terms $\xi_f dN_t$, Figure 4 compares the estimated jump probabilities to the ratios $\frac{\Delta F_t / F_t}{\sqrt{V_t}}$. These ratios are similar to the Z-statistics of Lee and

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Note that discontinuities in $\Delta F_t$, i.e., jump occurrences, are unobserved in discrete data. Therefore, we report probabilities of jump occurrences $P(N_{t+1} - N_t > 0 | F_i, i = 1, .., T)$ for each day $t$. 

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Mykland (2007) that enter the non-parametric Lee-Mykland test for jumps. The Z-statistics are approximately normally distributed in the absence of jumps under infill asymptotics, i.e., as the frequency of observations increases. Figure 4 plots $\frac{\Delta F_t/F_t}{\sqrt{V_t}}$, to which we also refer as Z-statistics, and marks days with an estimated probability of jumps above 20%. It follows from Figure 4 that the large negative values of the Z-statistics, below the 0.5th percentile of the standard normal distribution (-2.57), always correspond to estimated jump probabilities of more than 20%. The opposite relation also holds: the days on which we estimate the probability of a jump to exceed 20% typically correspond to Z-statistics that are around or less than $-2.57$. Note that the ratios $\frac{\Delta F_t/F_t}{\sqrt{V_t}}$ compare stock movements to the current level of volatility. Therefore, for the market return, extreme events captured by the terms $\xi_t^f dN_t$ in the SVCJ model correspond to large negative price movements that are inconsistent with the current level of volatility. In contrast, during the recent financial crisis, large absolute returns were primarily driven by high levels of volatility: these extreme returns are not detected as jumps and are best explained by continuous dynamics.

**Value.** Perhaps the most interesting feature of our analysis of the value factor is that the jumps in returns are typically positive and even make up a substantial contribution to its average returns. We estimate the total return to the value factor to be approximately 0.01% per day (2.52% annualized) of which we find approximately 30% attributable to rare, large positive jumps. Jumps seem to occur less frequently in the value factor than they do in the market return. There are only two days with estimated jump probabilities greater than 80%. One such episode, when the value factor increases by 1.60%, coincides with the (negative) market correction on July 7, 1986. Hence there is some, albeit informal, evidence of co-jumps in the market return and the value factor.

In addition to typically positive jump sizes for the value factor, we also find no evidence of a leverage effect for continuous shocks. The corresponding correlation is slightly positive yet very close to zero ($\rho = 0.037$). Because the value factor is constructed as a zero net cost position, this suggests that any correlation between continuous shocks to variance and continuous shocks to returns in the long end of the portfolio are offset by an equivalent relationship in the short end. The value factor is also less volatile, with a total variance of 0.29 compared to 1.09 for the
market return, and the volatility-of-volatility is also lower at \( (\sigma^J)^2 = 1.92 \) compared with 3.67 for the market. The same holds true for jump days: we estimate the variance of jump sizes in the value factor to be 1.9 compared to 3.7 for the market return. Volatility dynamics for the value factor and the market are similar, as follows from Figure 2, but do not completely coincide. For example, the rise in value factor volatility during the period 1998-2002 was almost as high as in 2008-2009. For the market return, on the other hand, the volatility in the latter period greatly surpasses that during 1998-2002.

We investigate the nature of jumps in the value factor by comparing the volatility jump markings in Figure 2, the estimated jump probabilities in Figure 3, and the Z-statistics in Figure 4. We conclude that the estimated jumps in the value factor are typically associated with an abnormal rise in volatility but not with abnormal returns.

**Momentum.** The momentum factor, much like the market return, is subject to infrequent \((\lambda = 0.0058)\) but large \((-1.68\%\) on average\) negative jumps. The total excess return on the momentum factor is 10.9\% annualized, after accounting for average annual losses due to jumps of \(-2.3\%\). There are five trading days for which the probability of a jump in the momentum factor exceeds 0.8. One such instance occurs on January 3, 2001, when the Federal Reserve announced a cut in the federal funds rate and technology stocks (NASDAQ) surged up by a record 14\% amid a prolonged bear market for the technology sector. On this date the momentum factor lost 7.49\%. In general, our univariate analysis yields no evidence of co-jumps in the market and momentum factors, e.g., January 3, 2001 is not a jump day for the market return.

In regard to the interpretation of jumps in the momentum factor, we find that many but not all jumps are associated with an increase in volatility. Similar to the market return, we observe a correlation between large negative (below \(-2.57\)) Z-statistics \(\frac{\Delta F_t/F_t}{\sqrt{V_t}}\) and high (above 20\%) estimated jump probabilities. We also find that jumps make up the largest fraction of total variance for momentum returns. As was the case for the value factor, we find no evidence of a leverage effect in the momentum portfolio. In fact, our estimate of \(\rho\) indicates a positive relationship between continuous variance shocks and shocks to the momentum returns. Comparing volatilities for the factors in Figure 2, we again find similarities in the dynamics of the momentum volatility and the market volatility. However, the dynamics clearly do not coincide.
For example, momentum volatility was largely unresponsive to the Asian crisis.

The univariate analysis in this section confirms similarities in the factor dynamics. The jump-diffusion SVCJ model captures the presence of extreme returns and extreme volatility changes in the market, value, and momentum factors.

3 Multivariate Model

Before we formulate a reduced-form model for the factors’ joint dynamics, we consider model components that we wish to examine with our multivariate analysis. We focus on the two main components of the joint model: standard correlations, as measured by the covariance matrix of continuous shocks, and extreme-event correlations, as measured by the characteristics of co-jumps.

It is important to note that univariate jump-diffusion models allow for a division of shocks into the following four categories: moderate-size shocks to volatility, moderate-size shocks to returns that are not associated with shocks to volatility, extreme shocks to volatility, and extreme shocks to returns that are not associated with extreme shocks to volatility. In contrast to the univariate case, the joint modeling of the market return with the other factors allows for further decomposition. As such, it is interesting to consider how the factors respond to each other’s volatility shocks and what the relationship exists between the remaining parts.

In regard to jump dynamics, the asset pricing literature provides some insights into possible jump co-interactions. One source of jumps in prices can be large changes in macroeconomic fundamentals (see, Drechsler and Yaron, 2011), in macroeconomic forecasts or macroeconomic uncertainty (see Shaliastovich, 2011). For this class of extreme events, we can expect co-jumps across factors of correlated sizes in returns and volatilities. Bansal and Shaliastovich (2011) identify an additional source of jumps in equity prices. They argue that jumps in financial markets come from unevenly paced investment research efforts. This endogenous, time-varying learning process triggers revisions in prices across assets. In general, this second class of co-jumps can be characterized by coincident timing, caused by a common initiating process. However, revisions in prices can be independent. Our model accommodates both classes of co-jumps described in this section.
Our first step in constructing the multivariate model is to specify the marginal dynamics of the factors \((F_1,t, F_2,t, F_3,t)\) following a template similar to Eraker, Johannes, and Polson (2003):

\[
\begin{align*}
\frac{dF_{i,t}}{F_{i,t}} &= \mu_{i,t-} dt + \sqrt{\nu_{i,t}} dB_{i,t} + \xi_{i,t}^f dN_{i,t}, \\
\ dv_{i,t} &= \mu_{v,i,t-} dt + \sigma_{i,v} \sqrt{v_{i,t}} dB_{i,t} + \xi_{v,i,t}^v dN_{v,i,t}, \quad i = 1, 2, 3,
\end{align*}
\]

(2)

where \(N_{i,t}\) and \(N_{v,i,t}\) are counting processes, \(\mu_{i,t-}\) and \(\mu_{v,i,t-}\) are predictable means of returns and volatilities, respectively. Index \(i = 1\) refers to the market return, index \(i = 2\) is for the momentum, and \(i = 3\) is for the value factor. Next we discuss how the above model accommodates a variety of possible cointeractions among factors.

**Model for volatilities.** Systematic volatility in this model is driven by three processes, \(V_{i,t}, i = 1, 2, 3\), wherein for simplicity we define \(V_{1,t}\) as the spot volatility of the first factor (market) and \(V_{2,t}\) and \(V_{3,t}\) as those of the momentum and value factors, respectively. As in the univariate case, we model expected changes in volatility as linear functions:

\[
\mu_{v,i,t-} = \kappa_{i,1}(\theta_1 - V_{1,t-}) + \kappa_{i,2}(\theta_2 - V_{2,t-}) + \kappa_{i,3}(\theta_3 - V_{3,t-}), \quad i = 1, 2, 3.
\]

(3)

The new feature is that we allow each factor’s volatility to be driven by all three volatility processes. This specification is equivalent to a model with three underlying autoregressive processes that affect volatilities of all factors. When analyzing estimation results, we return to this interpretation of the model and consider the relationship between these underlying components and factor returns.

**Model for diffusion shocks.** The covariance matrix of Brownian shocks \((dB_{1,t}, dB_{2,t}, dB_{3,t}, dB_{v,1,t}, dB_{v,2,t}, dB_{v,3,t})\) takes on the most general form. Corresponding correlations are denoted by \(\rho_{i,j}, i, j = 1, \ldots, 6\). For example, \(\rho_{1,5}\) is the correlation between the first \(dB_{1,t}\) and the fifth \(dB_{v,2,t}\) shocks.

**Model for jumps.** Because large jumps occur rarely, accurate estimation of the co-jump dynamics is challenging without some restrictions on the jumps’ higher moments and other aspects of their dynamics. In prior versions of this paper, we estimated a co-jump model with an unrestricted jump size covariance structure. Using indirect measures of estimation quality, such as the stability of MCMC chains, we concluded that such estimation cannot proceed without
excessively strict priors on parameters. Therefore, we proceed by imposing further parametric structure on jump cointeractions.

In the choice of the jump model, we are guided by the following considerations. First, we note that factors’ volatilities are correlated; in particular, Figure 2 suggests a common volatility component. Therefore, our model includes a volatility channel for the transmission of extreme events, which we refer to as the volatility jump beta. Second, we include a similar transmission mechanism for extreme returns, which we refer to as the return jump beta. Return jump betas lead to correlation in the magnitude of extreme events across assets, as it happens in Drechsler and Yaron’s (2011) model. Third, we allow extreme events in the market to carry over to the other factors without co-movement in returns. Under this scenario, factor jumps are due to a common initiating process; however, the jump sizes are uncorrelated as in Bansal and Shaliastovich’s (2011) model. Fourth, as in the SVCJ model of Eraker, Johannes, and Polson (2003), we require the return and volatility of each factor to jump simultaneously. Lastly, we allow for an additional source of risk in momentum and value strategies by including individual jumps in those factors. The presence of individual jumps is suggested by our univariate analysis in Section 2.

Our final model, which accommodates all of these features, is constructed as follows. We begin with the model for market jumps. Following Eraker, Johannes, and Polson (2003) we restrict $N_{1,t} = N_{1,t}^v$ to be a Poisson process with intensity $\lambda_1$ and model $\xi_{1,t}^f$ and $\xi_{1,t}^v$ as conditional normal and exponential random variables, respectively. The model for the jump process in the $i$th factor, where $i$ is either 2 or 3, is decomposed as follows:

$$
\xi_{i,t}^f dN_{i,t} = \beta_i^f \xi_{1,t}^f dN_{1,t} + \xi_{i,t}^{f,c} dN_{1,t} + \tilde{\xi}_{i,t}^f d\tilde{N}_{i,t},
$$

$$
\xi_{i,t}^v dN_{i,t} = \beta_i^v \xi_{1,t}^v dN_{1,t} + \xi_{i,t}^{v,c} dN_{1,t} + \tilde{\xi}_{i,t}^v d\tilde{N}_{i,t},
$$

(4)

where $\tilde{N}_{2,t}$ and $\tilde{N}_{3,t}$ are Poisson process with intensities $\lambda_2$ and $\lambda_3$, independent of $N_{1,t}$. The first components, $\beta_i^f \xi_{1,t}^f dN_{1,t}$ and $\beta_i^v \xi_{1,t}^v dN_{1,t}$, are co-jumps with the market due to jump betas. The second components, $\xi_{i,t}^{f,c} dN_{1,t}$ and $\xi_{i,t}^{v,c} dN_{1,t}$ are co-jumps with the market due to a common process that causes jumps in all factors. The third components are individual jumps with sizes $(\tilde{\xi}_{i,t}^f, \tilde{\xi}_{i,t}^v)$ that are also modeled by conditional normal and exponential variables, respectively.

We retain the possibility of individual jumps in the market that are not reflected in the
other factors by setting \((\xi_{i,t}^f, \xi_{i,t}^v)\) to zeros \(^8\) with probability \(1 - \lambda_{i,c}\). For parsimony, we set the distribution of \((\xi_{i,t}^f, \xi_{i,t}^v)\) to that of \((\tilde{\xi}_{i,t}^f, \tilde{\xi}_{i,t}^v)\) in the rest of the cases. \(^9\) That is, the distribution of \((\xi_{i,t}^f, \xi_{i,t}^v)\) is a mixture of the distribution of \((\tilde{\xi}_{i,t}^f, \tilde{\xi}_{i,t}^v)\) with probability \(1 - \lambda_{i,c}\) and the distribution of the constant \((0, 0)\) with probability \(\lambda_{i,c}\), \(\lambda_{i,c} \in [0, 1]\). Thus, \(\lambda_{i,c}\) is the probability of a co-jump, apart from those due to jump betas \((\beta_{i,J}^f, \beta_{i,J}^v)\).

To summarize, jump dynamics can be described as follows. (i) Market return and market volatility jump simultaneously. (ii) A jump in market return can affect the other factors with sensitivities determined by jump betas, \(\beta_{i,J}^f\). (iii) A jump in the market volatility can affect the volatility of the other factors with sensitivities determined by volatility jump betas, \(\beta_{i,J}^v\). (iv) Jumps in market return can trigger jumps in the other factors with probabilities \(\lambda_{i,c}\). (v) The other factors can jump independently of market return.

The estimation is implemented using the following discretization scheme:

\[
\frac{F_{i,t} - F_{i,t-h}}{F_{i,t-h}} = \mu_1 h + \sqrt{V_{i,t-h}} h \varepsilon_{1,t} + \xi_{i,t}^f J_{1,t},
\]

\[
\frac{F_{i,t} - F_{i,t-h}}{F_{i,t-h}} = \mu_i h + \sqrt{V_{i,t-h}} h \varepsilon_{i,t} + \beta_i^J \xi_{i,t}^f J_{1,t} + \xi_{i,t}^f J_{i,t}, i = 2, 3,
\]

\[
V_{i,t} - V_{i,t-h} = \mu_i^v h + \sigma_{i,v} \sqrt{V_{i,t-h}} h \varepsilon_{i,t}^v + \xi_{i,t}^v J_{i,t},
\]

\[
V_{i,t} - V_{i,t-h} = \mu_i^v h + \sigma_{i,v} \sqrt{V_{i,t-h}} h \varepsilon_{i,t}^v + \beta_i^v \xi_{i,t}^v J_{i,t} + \xi_{i,t}^v J_{i,t}, i = 2, 3,
\]

where \((\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}, \varepsilon_{1,t}^v, \varepsilon_{2,t}^v, \varepsilon_{3,t}^v)\) are zero-mean, jointly normal, random variables with unit variances on the diagonal of the covariance matrix. The extreme market shocks \((\xi_{i,t}^f, \xi_{i,t}^v)\) have the same distribution as \((\xi_{1,t}^f, \xi_{1,t}^v)\) and the shocks \((\xi_{i,t}^f, \xi_{i,t}^v)\), where \(i\) is either 2 or 3, have the same distribution as \((\tilde{\xi}_{i,t}^f, \tilde{\xi}_{i,t}^v)\). The processes \(J_{i,t}\), \(J_{2,t}\) and \(J_{3,t}\) are Bernoulli variables \(^{10}\) with parameters \(\lambda_{i,h}, 1 - (1 - \lambda_2 h)(1 - \lambda_v J_{1,t})\) and \(1 - (1 - \lambda_3 h)(1 - \lambda_v J_{1,t})\), respectively. The estimation is

\(^8\)Note that this model can be represented as a specification with five jump processes: three individual jump processes for market, momentum, and value returns, and two co-jump processes for the market-momentum and market-value pairs. To obtain this specification, \(N_{1,t}\) is divided into the market individual jump \(\tilde{N}_{1,t}\) and two co-jumps \(N_{i,t}^{c,i}, i = 2, 3\), for which the division happens according to whether \((\xi_{i,t}^f, \xi_{i,t}^v) = (0, 0)\) or \((\xi_{i,t}^f, \xi_{i,t}^v) \neq (0, 0), i = 2, 3\).

\(^9\)The same assumption implicitly holds in the model with self-excitation by Ait-Sahalia et al. (2014). In that model, the jump in Asset A can be "excited" by prior jumps in Asset B, or can arise in the absence of jumps in Asset B. Regardless of prehistory, the distribution of jump size remains the same.

\(^{10}\)Note that we conveniently regrouped jump components. However, we estimate the parameters of the original model. The proposed discretization scheme excludes the probability of a double jump in the secondary factors \((i = 2, 3)\) due to simultaneous realizations of \(\xi_{i,t}^f dN_{1,t}\) and \(\tilde{\xi}_{i,t}^f dN_{i,t}\) components.
performed using the MCMC method with $h = 1$, i.e., in daily units. The details of the MCMC method are presented in the supplementary materials for this paper.

4 Estimation Results

4.1 Model Selection on Pairwise Models

Increasing the number of series in the SVCJ model leads to estimation problems that are commonly known as the curse of dimensionality. In particular, we observe that the estimation quality of jump size correlations deteriorates with the increase in the number of factors. The MCMC chains of the jump-related parameters become less stable as we increase the number of factors from two to three. Therefore, we performed a model pre-selection step for co-jump dynamics on pairwise bi-variate models for market-momentum and market-value pairs.

In model (4), the jump size correlations are captured by jump betas $\beta_{i,J}^{f,J}$ and $\beta_{i,J}^{v,J}$. We performed a formal Bayesian model selection for the choice of co-jump specifications as in George and McCulloh (1995), Carlin and Chib (1995), and Carlin and Polson (1991). We found that we should prefer a model with co-jumps to a model without co-jumps. The corresponding Bayes factors took values 5050 and 3366 for the market-momentum and the market-value pairs. A model with correlated jump sizes in volatility should possibly be preferred to a model with uncorrelated volatility jump sizes. The corresponding Bayes factors were 4.5 and 5.2 for the market-momentum and the market-value pairs. Finally, a model with correlated jumps in returns does not improve significantly on the previous specification. The corresponding Bayes factors were 2.6 and 2.2 for the market-momentum and the market-value pairs, respectively. When interpreting the Bayes factors, we followed the guidelines of Kass and Raftery (1995) who interpret a Bayes factor greater than 150 as decisive evidence in favor of a better fitting model. A Bayes factor greater than 20 is interpreted as a strong evidence, and greater than 3, as positive evidence. Therefore, based on this metric, one should somewhat prefer a model with correlated jumps in volatility. However, the estimated volatility jump betas $\beta_{i,J}^{v,J}$ appeared rather small at 0.154 for the momentum and 0.130 for the value factor. Therefore, for our final results, we restrict all jump size correlations to zero to improve the stability of the MCMC estimation for the tri-variate system.
4.2 Estimation Results

Table 4 reports the parameter estimates. Table 5 reports the corresponding derived statistics that allow us to interpret some estimates in Table 4. We divide the discussion of these results into several paragraphs.

**Implications for the interrelation of volatilities.** Correlations between continuous shocks to volatilities are estimated at 0.89 with a standard deviation (s.d.) of 0.03, at 0.88 (s.d. 0.02), and at 0.93 (s.d. 0.02) for the market-momentum, market-value, and momentum-value pairs. We confirm that factor volatilities are correlated but do not entirely coincide. Thus, a model with several positively correlated volatility components is more appropriate.

We proceed with the spectral decomposition of the matrix $I - \kappa$. We find that this matrix has two roots that are statistically the same, approximately 0.99, and one root that is smaller, 0.966. We next consider the slow and fast varying volatility components that are obtained as linear combinations of $V_{i,t,j} = 1, 2, 3$, with the slowest and fastest decaying autocorrelations, respectively. The variance decomposition of continuous shocks shows that 83% of momentum factor variance is due to the slow-varying component (s.d. 6%). Similarly, for the value factor, 79% (s.d. 9%) is due to the slow-varying component. For market volatility, the division is less polarized: 62% is due to the fast-varying volatility and 25% is due to the slow-varying volatility, while the rest is due to the covariance between the components. Therefore, the momentum and value volatilities convey information about the slow component of the market volatility $V_{1,t}$. Further analysis suggests that the momentum volatility is a better measure of this component. For example, the momentum volatility Granger-causes $V_{1,t}$, while the value volatility does not, which follows from the estimates of $\kappa_{1,2}$ and $\kappa_{1,3}$ reported in Table 4.

**Implications for the diffusion leverage effect.** It is common to refer to the negative correlation between returns and shocks to volatility as the leverage effect. Though initially explained by the increased leverage of companies with decreasing stock prices (see Black, 1976, and Christie, 1982), it is now more prevalent to interpret the leverage effect as coming from an increase in risk premia demanded by investors when volatility is high (see French et al., 1987, Campbell and Hentshel, 1992). We interpret the slow and fast-varying volatilities constructed

\[11\text{Matrix } \kappa \text{ is a } 3 \times 3 \text{ matrix of mean-reversion coefficients in the model of volatilities (3)}\]
from $V_{1,t}$, $V_{2,t}$, and $V_{3,t}$ as two different sources of risk and analyze the leverage effect for each factor with respect to each of these components.

Panel B in Table 5 shows return correlations with the slow-varying volatility, and the partial correlations with the fast-varying volatility controlling for the effect of the slow component. We find that the negative leverage effect with respect to the slow volatility component explains 8.5% of the market continuous shocks, while the fast volatility component explains 20.5% of the market continuous shocks. The fast-varying component has a noticeably larger effect, which agrees with Bollerslev, Sizova, and Tauchen (2011), who found that the faster-varying component of the market volatility is the main driver of the leverage effect at high frequencies. For momentum, only the faster-varying component produces a negative leverage effect. This component explains 9.0% of the momentum shocks. In contrast, the slow-varying component results in a small and positive leverage effect. For the value factor, none of the volatility components produce a negative leverage effect.

Finally, we report the estimates of the total return variation ($R^2$) explained by the leverage effect. Correlations with volatilities explain 31.7% of the continuous shocks to market returns, 14.5% of the continuous shocks to momentum returns, and only 2.4% of the continuous shocks to value returns. We conclude that the leverage effect is less important for the other two factors than for the market return. Because the diffusion leverage effect reflects adjustment in prices in response to the increased diffusion risk, we conclude that this type of risk is irrelevant for the value factor and has a complex structure for the momentum. The momentum returns are negatively affected by the fast volatility component and are positively affected by the slow volatility component.

**Jump leverage effect.** The leverage effect in the jump component is a less commonly studied phenomenon, for which Drechsler and Yaron (2011) provide some theoretical discussion. Under their assumptions, jumps in volatility reflect negatively (with perfect correlation) in the market return. In our model, this leverage effect can be captured by a negative $\rho^J$. As follows from the estimation results in Table 5, the estimated $\rho^J$ for the market and momentum factors explain a small positive volatility effect is explained by the negative correlation between the market and value returns. Value returns are positively correlated with volatility shocks because market returns are negatively correlated with the same shocks.
are negative $-4.7$ and $-4.8$, respectively. As follows from the estimation results in Table 5, the estimated $\rho^J_i$ for the value factor is positive.

We also suggest another measure of the jump leverage, which appears more robust to the choice of the model. This measure is based on the asymmetry of extreme returns. Consider a simple example in which volatility jumps $\xi^v_{i,t} > 0$ and corresponding return cojumps $\xi^f_{i,t} = \rho^J_i \xi^v_{i,t}$, $\rho^J < 0$, are constant. The correlation between these jumps is zero, but the leverage effect still manifests itself in the existence of co-jumps between the volatility and returns with a negatively skewed distribution of extreme returns. Therefore, we propose to infer the sign of the jump leverage effect by the sign of the expected extreme returns, which are reported in Panel C of Table 5. As follows from the table, both the market and momentum exhibit negative asymmetries during extreme events. We estimate the expected jump size of the market to be $-0.75\%$. For momentum, we estimate the expected jump size to be $-1.18\%$. The expectation of the jump size in the value factor is small and positive. We conclude that there is a negative jump leverage effect in both the market and momentum factors. We find no evidence of a jump leverage effect in the value factor. Again, linking the presence of the jump leverage effect to the presence of the premia paid on jump shocks, we conclude that part of the market and momentum and market premia can be due to the jump risk.

**Implications for co-jumps.** Next, we revisit the empirical findings regarding co-jumps. In the prior univariate analysis of Section 2, we found only one co-jump for the market-value pair and none for the market-momentum pair. In our multivariate analysis, we now find that most market jumps are associated with jumps in momentum. The probability of a co-jump is $\lambda_c = 0.96$ (s.d. 0.04). Similar results are obtained for the value factor, evidenced by the estimate of $\lambda_c = 0.84$ (s.d. 0.09).

**Implications for individual jumps in momentum and value.** Finally, we answer the question: do we observe jumps in momentum and value returns that are not associated with a jump in the market return? These individual jumps occur with intensity $\lambda_2$. For the momentum factor, the estimate of $\lambda_2 = 0.002$ (s.d. 0.001) implies that there are approximately 16 individual momentum jumps in the sample. The same estimate for the value factor is both small ($\lambda_2 = 0.0007$) and statistically insignificant (s.d. 0.0005). Thus, we do not find conclusive
evidence of independent jumps in the value factor in the absence of extreme market returns.

To summarize our findings on extreme events affecting momentum and value factors and their co-interaction with market returns, we conclude that the behavior of the momentum and value are markedly different. Value returns do not experience additional extreme events and only respond to extreme changes in the market returns. Furthermore, value returns exhibit neither continuous, nor jump leverage effects. The momentum returns jump occasionally independent of the market returns. The leverage effect associated with momentum returns features a complex structure: only the fast and jump components of volatility produce negative effect on momentum returns.

5 Implications for Factor Investing

From the equilibrium point of view, the optimal portfolios of investors should include the market asset alone. However, a portfolio selection exercise under a simple preference structure can provide insight into the risks associated with investment in momentum and value strategies. In this section, we apply estimation results of Section 4 to construction of optimal portfolios. The analysis in this section should be interpreted with caution because it is, certainly, affected by the choice of the reduced-form model (2) and investor preferences.

It is well documented that factor strategies deliver positive average excess returns without exposing investors to excessive risk. Likewise, this feature of momentum and value strategies is observed in our data set. As an illustration, we evaluate the performance of a portfolio of factors constructed by a risk-averse CRRA investor who re-balances her portfolio daily to maximize a myopic utility function. For each day $t$, the investor maximizes expectation of the next-day utility

$$
E_t \left( \frac{W_t R_{t,t+1}^{1-\gamma}}{1-\gamma} \right), \quad R_{t,t+1} > 0,
$$

where $W_t$ is the accumulated wealth at time $t$ normalized to one dollar at the beginning of the sample, $R_{t,t+1}$ is the total return, and $\gamma > 0$ is the risk aversion coefficient. If $R_{t,t+1} \leq 0$, then the utility is set to $-\infty$. That is, the investor is infinitely averse to losing all of the wealth.

13 This finding implies that the long and short positions of the value factor adjust to the volatility shocks in the same way.
The investor allocates her wealth across the market index, a risk-free bond whose return is approximated by the Treasury bill rate in daily units, and either momentum or value strategies\(^{14}\). The corresponding dynamic weights \(w_{1,t}, (1-w_{1,t}), \text{ and } w_{2,t}\) are optimally chosen at the beginning of each day. Note that due to the possibility of unlimited daily losses under our reduced-form model, the optimal positions in equities must be exactly zero \(^{15}\), i.e., \(w^*_{1,t} = 0\) and \(w^*_{2,t} = 0\). To avoid this trivial result, we restrict maximum daily losses in factor returns to \(-33\%\), which is a conservative lower bound \(^{16}\) as follows from Figure 1. No restrictions are imposed on the upside risk. Consequently, the optimal portfolio weights must satisfy conditions \(0 \leq w^*_{1} \leq 3\) and \(0 \leq w^*_{2} \leq 3\). Additionally, we impose a restriction on the total position to capital as in Christoffersen and Langlois (2011). With a margin requirement of 20\%, this restriction becomes \(w^*_{1} + 2w^*_{2} \leq 5\).

The portfolio optimization is performed using simulations. Hypothetical trajectories of the returns are simulated using \(SVCJ\) model with parameters estimated over the entire sample. The unobserved states at time \(t\), i.e. their posterior distribution, are estimated using the available data at time \(t\). Most importantly, \(\text{we do not rely on the assumption that the average factor premia over the whole sample are known}\) to the investor. Instead, we model the premia as slow-varying processes whose values at time \(t\) can be only estimated using information at time \(t\). In particular, we specify intercepts \(\mu_{i,t}\) in model (2) as follows:

\[
d\mu_{i,t} = \sigma_{\mu_i} dB_{\mu_i,t}, \quad i = 1, 2, 3,
\]

where \(B_{\mu_{1,t}}, B_{\mu_{2,t}}, \text{ and } B_{\mu_{3,t}}\) are independent Brownian processes. Therefore, the expected excess returns follow a random walk model at discrete times. A similar approach to time variation in coefficients can be found in many applications across disciplines (see Durbin and Koopman,\(^{17}\)).

\(^{14}\)That is, we consider two strategies separately: one includes the momentum factor and the other includes the value factor.

\(^{15}\)This result follows from the arguments by Liu, Longstaff, and Pan(2003) as in Theorem 1 with \(X_{\sup} = +\infty\) and \(X_{\inf} = -\infty\). That is, if the equity price paths are discontinuous without bounds on instantaneous losses, then the investor faces a risk of the negative wealth before she has a chance of re-balancing her portfolio. Therefore, any non-zero equity position brings her utility of \(-\infty\). Note that this is not the case for continuous price dynamics with continuous portfolio re-balancing. Liu, Longstaff, and Pan(2003) liken this risk of large instantaneous losses to the risk from holding illiquid assets.

\(^{16}\)The results of this section are not sensitive to the choice of the bound as long as the daily returns are separated from -100\%. With parameter values reported in Table 4, the chances of such large negative returns are sufficiently small not to affect expectations \(\mathbb{E}_t \left(\frac{(W_t R_{t+1})^{1-\gamma}}{1-\gamma}\right)\). Results in Table 7 do not impose bounds on daily losses.

Because the premia are now latent, they are estimated together with the other unobserved states, such as volatilities. Figure 5 shows the premia\(^{17}\) estimated using the entire sample and the same premia estimated using only information up to time \(t\). As expected, there is a delay in the latter series. The most important finding from Figure 5 is that the market, momentum, and value strategies differ remarkably in the stability of their premia. The market premium is the most stable, while the value premium is the most volatile, with periods of negative values. We conclude that premium uncertainty is an important source of risk for the value strategy.

Once the optimal weights are determined, our first result is to document the characteristics of the corresponding returns. The benchmark portfolio includes only the market index and risk-free bond. As expected, adding momentum or value strategies to the portfolio mix allows an investor to significantly outperform the benchmark. The difference in certainty equivalents between the market-bond strategy and momentum-market-bond strategy is 10.9% annualized (\(t\)-statistic of \(t = 1.9\)) for \(\gamma = 2\) and 6.9% annualized (\(t = 2.4\)) for \(\gamma = 10\). The same differences for the value-market-bond strategy are 11.6% (\(t = 3.8\)) and 8.2% (\(t = 4.7\)) annualized for \(\gamma = 2\) and \(\gamma = 10\), respectively. The results for the other values of \(\gamma\) are reported in Table 6. Based on Table 6, the positive gain in certainty equivalent is preserved over a range of \(\gamma\). More risk averse investors invest less in equities, which reduces the certainty equivalents of all equity strategies.

Our further analysis focuses on the extreme event risk. Figure 6 illustrates how the portfolios described above differ in their exposures to this risk. In this exercise, extreme events are proxied by the days with the highest market volatility. In particular, as follows from Figure 6, market volatility levels above 1.5 occur during turbulent periods that include many volatility jumps across factors. To capture the non-linear relationship between portfolio performance and volatilities, we consider days with estimated volatilities \(V_{1,t}\) above a range of thresholds \(V_1\) and estimate annualized certainty equivalents (CE) for CRRA preferences with \(\gamma = 2\) of the corresponding optimal factor strategies for these days only. Figure 6 plots these CEs against the thresholds \(V_1\). Gray areas mark 90% confidence intervals calculated using block bootstrap. As

\(^{17}\)The premia are calculated as \(\mu_{i,t}\) less the expected loss due to jumps.
expected, higher volatility negatively affects the performance of all strategies, with positive CEs eventually turning into zeros at $V_1$ between 0.7 and 1.0. However, the most dramatic volatility effect is seen for momentum. For the momentum strategy, CE becomes negative for volatilities above 1.2. Bad performance of momentum after a period of volatile returns was documented by Daniel and Moskowitz (2014), who explain this characteristic by the embedded option in momentum returns. Our result is similar in that we find a negative relationship between volatility and momentum gains. However, Daniel and Moskowitz (2014) dismiss the role of extreme event risk in the found phenomenon, pointing to the long duration of periods during which recent “losers” outperform recent “winners.” To assess the role of extreme events for the pattern in Figure 6, we repeat our calculations after removing all of the days with an estimated probability of jumps above 20%. The resulting CEs are plotted with dashed lines. We find the performances of all portfolios to be improved. However, these improvements are not statistically significant for market and value strategies, while they are dramatic for momentum. Thus, in contrast to the findings of Daniel and Moskowitz (2014), we document the important role of volatility jumps for momentum strategy risks.

A more detailed analysis of the portfolio performances during extreme events is complicated due to the rarity of salient jump events. Because extreme events happen rarely, back-tests with only such days cannot be accurate. Alternatively, the jump-diffusion model presented in this paper readily yields estimates of average extreme returns and other moments.

Characteristics of the returns on jump days are summarized in Table 7. We expect two effects. On one hand, adding additional factor strategies to a market portfolio increases the occurrence of jump days to the total frequency of $\lambda_1 + \lambda_i, i = 2, 3$. On the other hand, due to the zero correlation between jump magnitudes of returns, i.e., $\beta_{f,J} = 0$, additional factor strategies diversify the risk of the portfolio on market jump days. To evaluate this trade-off, we study the following moments of the returns: means, variances and VaRs, on days on which at least one of the factors in a pair jumps. Table 7 reports the results for different choice in weight ratios $w_2/(w_1 + w_2) \in [0, 1]$, where $w_1$ is the contribution of the market index and $w_2$ is the weight of the additional factor strategy. The weights are chosen so that the average premium remains equal to the market premium.
For mean returns, we find that the momentum component increases average losses during extreme events from $-0.68\%$ per event for $w_2 = 0$ to $-1.14\%$ per event for $w_1 = 0$. This finding, albeit statistically insignificant, again points to the presence of additional extreme event risk in the momentum strategy. For the value strategy, a mean loss during extreme events gradually decreases as we diversify away from the market portfolio. Thus, we find no evidence of extreme event risk in the value portfolio.

In regard to variances, we find that adding a momentum strategy reduces the risk of the portfolio during jump days as $w_2/(w_1 + w_2)$ increases from 0 to 0.6. The variance of the portfolio does not significantly change for $w_2/(w_1 + w_2)$ from 0.6 to 0.8, and increases as we go from $w_2/(w_1 + w_2) = 0.8$ to 1.0. Similarly, VaRs repeat the same pattern. This evidence, taken together with results on the average extreme returns, indicates that a momentum strategy reduces extreme Value-at-Risk, though historically it has led to a larger expected shortfall. In contrast to momentum strategy, adding a value strategy to the market return simply diversifies the portfolio; the average losses per extreme event, variance, and the VaR are all reduced.

To summarize, combining factors in one portfolio historically reduced the extreme-event risk measured by volatilities and Value-at-Risk. However, we find an increased risk stemming from the momentum strategy when this risk is measured in other ways. First, a portfolio that includes the momentum factor dramatically underperforms during extreme volatility periods. We find this phenomenon to be attributable to the jump risk. Second, we document higher average losses during extreme events from the portfolio that includes the momentum factor. In contrast to the momentum factor, there is no additional jump risk in the value strategy. However, value strategy features unstable and often negative premium.

## 6 Do Jumps Explain Premia?

In the previous section, we found evidence of the increased jump risk in the momentum strategy. In this section, we would like to test if the momentum and value factors carry seemingly large risk premiums because they are portfolios that have some increased tail risk exposure. For this, we consider a framework in which the investor wealth comes only from investing in the market return. With our estimation results, we can address the following two questions. First, what
is the order of magnitude of the jump risk premium in the moment and value factors? This question is currently being addressed for the market return using the information contained in option prices on broad stock market indices\footnote{While options on broad stock market indices can be considered as proxies to the options on the market return, options on the momentum and value factors do not exist.}, see Liu, Pan, and Wang (2005), Xu (2016), and Seo and Wachter (2016). With our empirical analysis, we can assess the risk jump premium carried by the momentum and value strategies using the estimated model of the joint factor dynamics. Second, what is the order of magnitude of the rare event premia due to rare event model uncertainty? The rare event premium was considered by Liu, Pan, and Wang (2005) and Drechsler (2013) in a robust-control framework, in which investors maximize their worst-case scenario utilities. With our model, we can give an estimate of the rare event premia for the factors assuming that the worst-case scenario is statistically indiscernible from our estimated model\footnote{We consider two models statistically indiscernible if their parameters are not more than two posterior deviations apart. The posterior standard deviations are common measures of the estimation accuracy in the Bayesian analysis.}.

### 6.1 Risk Premia

Consider the stochastic discount factor (SDF) that, in equilibrium, takes the following form:

\[
d\ln M_t = \gamma_0 dt - \gamma_1 d\ln R_t - \gamma_2 d\ln R_{\perp,t},
\]

where \(R_t\) is the return process for the aggregate stock market and \(R_{\perp,t}\) is an index of other factors that affect stock market valuations. Examples of SDFs that take the above form include the single-factor capital asset pricing model (CAPM) and general equilibrium model with Epstein-Zin preferences. For the CAPM, the last term is absent, i.e., \(\gamma_2 = 0\). For the Epstein-Zin preferences with only capital income, \(\ln R_{\perp,t}\) is the consumption growth, \(\gamma_1 = 1 - \theta\), where \(\theta = (1 - \gamma) \times (1 - 1/\psi)^{-1}\), \(\gamma\) is the risk aversion and \(\psi\) is the elasticity of intertemporal substitution, see Bansal and Yaron (2004) for the discrete-time case and Bollerslev, Sizova, and Tauchen (2012) for the continuous-time setting. The second loading is \(\gamma_2 = \theta \psi^{-1}\) and \(\gamma_0 = -\theta \delta\), where \(\delta\) is the time preference parameter.

If the other factors \(R_{\perp,t}\) in (6) have an unspecified co-jump relationship with the momentum and value returns, we can explain momentum and value premia of arbitrary sizes exploiting
different assumptions on $R_{\perp,t}$. To constrain our analysis, we impose restrictions on $R_{\perp,t}$ and only allow for $R_{\perp,t}$ that changes gradually in time, i.e., it is not subject to jumps. This assumption trivially holds for the CAPM, from which $R_{\perp,t}$ is absent. This assumption holds for the Epstein-Zin utility if, as argued by Bansal and Shaliastovich (2011), there are no abrupt changes in the observed aggregate consumption $^{20}$.

Under this restriction, the only source of the jump risk is co-jumps with the market return. Note that this restriction on $R_{\perp,t}$ is also an identifying restriction on $\gamma_1$, because $R_{\perp,t}$ and $R_{t}$ can be correlated.

Let the market return follow a general jump-diffusion dynamics:

$$
\frac{dR_t}{R_t} = (r_{t-} + \tilde{\pi}_{R,t-})dt + \sqrt{V_{t-}^R} dB_t^R + \xi_t^R dN_t^R,
$$

where $r_{t-}$ is the instantaneous risk-free rate, $V_{t-}^R$ is the spot volatility, $B_t^R$ is the standard Brownian motion, $N_t^R$ is a generalized Poisson process with intensity $\lambda_{R,t}$, and $\xi_t^R$ is the jump size. Under this model, the total instantaneous market premium is $\pi_{R,t-} = \tilde{\pi}_{R,t-} + \lambda_{R,t-} E_{t-} \xi_t^R$, where the second part is the expected loss due to extreme events. Under the no-arbitrage condition, the process $M_t R_t$ is a martingale. This, together with the dynamics of the SDF (6) and market return (7), gives the following expression for the instantaneous premium:

$$
\pi_{R,t-} = \gamma_1 V_{t-}^R + \gamma_2 \text{cov}_{t-} (R_t, R_{\perp,t}) + \lambda_{R,t-} E_{t-} \xi_t^R,
$$

where $\text{cov}_{t-} (R_t, R_{\perp,t})$ is the instantaneous continuous covariance between the market return and index of other factors. The last part of the premium is due to the jump risk. The proof of the above result is given in the appendix.

Now, consider an arbitrary asset with return dynamics

$$
\frac{dR_{i,t}}{R_{i,t-}} = (r_{t-} + \tilde{\pi}_{i,t-})dt + \sqrt{V_{t-}^i} dB_t^i + \xi_t^{i,c} dN_t^R + \xi_t^i dN_t^i,
$$

where the jump part is divided into the cojump part $\xi_t^{i,c} dN_t^R$ and an individual jump $\xi_t^i dN_t^i$. As before, $B_t^i$ is a standard Brownian motion, that is possibly correlated with $B_t^R$. The Poisson process $N_t^i$ has intensity $\lambda_{i,t}$. The spot variance is $V_{t-}^i$. The premium paid on this asset is $\pi_{i,t-} = \tilde{\pi}_{i,t-} + \lambda_{R,t-} E_{t-} \xi_t^c + \lambda_{i,t-} E_{t-} \xi_t^i$. The no-arbitrage condition yields the following decomposition of

$^{20}$ Jumps in the expected consumption growth and other conditional moments are permitted.
the premium:

\[ \pi_{i,t} = \gamma_1 \text{cov}_{t-} (R_{i,t}, R_t) + \gamma_2 \text{cov}_{t-} (R_{i,t}, R_{\perp,t}) + \lambda_{R,t-} E_{t-} \left[ \xi^c_{i,t} - (1 + \xi^R_t)^{-\gamma_1} \xi^i_{t-} \right]. \]

Suppose, the short position of a factor portfolio yields return \( R_{-,t} \) and the long position of the factor portfolio yields return \( R_{+,t} \). In our notations, excess return \( \text{d}F_t/F_t \) stands for the difference \( \text{d}R_{+,t} / R_{+,t} - \text{d}R_{-,t} / R_{-,t} \). The corresponding premium of the factor portfolio is equal to

\[ \pi_{f,t} = \pi_{+,t} - \pi_{-,t} = \gamma_1 \text{cov}_{t-} (F_t, R_t) + \gamma_2 \text{cov}_{t-} (F_t, R_{\perp,t}) + \lambda_{R,t-} E_{t-} \left[ \xi^f_{t-} (1 - (1 + \xi^R_t)^{-\gamma_1}) \right], \tag{9} \]

where \( \xi^f_t \) is the size of the jump in the factor return in response to the jump in the market.

We now state the following result which applies the above formulas to the SVCJ model estimated in this paper.

**Theorem 1.** Let the market return, momentum return, and value return follow the jump-diffusion dynamics given by system (2) with further restrictions imposed in Section 3. Suppose the stochastic discount factor is given by formula (6), where \( R_t = F_{1,t} \) and \( R_{\perp,t} \) is an index of unspecified additional factors with continuous diffusion dynamics. Then, the jump risk premium for market returns equals

\[ \lambda_1 E \left[ \xi^f_{1,t} (1 - (1 + \xi^f_{1,t})^{-\gamma_1} (1 + \xi^f_{1,t})^{1-\gamma_1}) \right], \]

where \( \xi^f_{1,t} \) is conditionally normal \( N \left( \mu^f_{1}, \rho^f_{1} \xi^v_{1,t}, [\sigma^f_{1}]^2 \right) \) and \( \xi^v_{1,t} \) is exponentially distributed \( \exp(\mu^v_{1}) \).

The average jump risk premia for secondary factors \( (i = 2, 3) \) equal

\[ \beta^f_{1,i} \lambda_1 E \left[ \xi^f_{i,t} (1 - (1 + \xi^f_{1,t})^{-\gamma_1}) \right] + \lambda_1 \lambda_{i,t-} E \left[ \tilde{\xi}^i_{t-} (1 - (1 + \xi^f_{1,t})^{-\gamma_1}) \right], \]

where \( \tilde{\xi}^i_{t-} \) is conditionally normal \( N \left( \mu^i_{1} + \rho^i_{1} \tilde{\xi}^v_{i,t}, [\sigma^i_{1}]^2 \right) \) and \( \tilde{\xi}^v_{i,t} \) is exponentially distributed \( \exp(\mu^v_{i}) \).

Figure 7 (solid lines) plots the jump premia in momentum and value as a function of the market jump premium. These graphs are obtained by varying the loading \( \gamma_1 \) between 1.5 and 20. We conclude that the momentum premium due to the jump risk is positive and approximately 30% of the jump premium in market returns. The value jump premium is negative because the value factor hedges the market jump risk.
6.2 Rare-Event Model Uncertainty Premia

We now consider the second source of the jump premium suggested by Liu, Pan, and Wang (2005). They hypothesize that the premia on extreme events is formed differently from how the premia on day-to-day shocks is generated. This difference stems from the very nature of extreme events that occur rarely. Therefore, there is uncertainty about the correct statistical model of extreme events. Liu, Pan, and Wang (2005) derived the jump premia on the market return in a general equilibrium model while assuming that investors maximize their utility under the worst-case scenario. As a result, equity premia include an additional part that is due to the uncertainty about the true $\lambda_{R,t}$ and true distribution of $\xi^R_{R,t}$.

Liu, Pan, and Wang (2005) allow a collection of possible models to be arbitrarily far from the true data-generating process, but they introduce an additional term in the utility of investors that penalizes deviations between the true model and worst-case scenario. For our analysis, it is natural to assume that the collection of possible models is defined by the accuracy of the model estimates. In particular, the worst-case scenario has parameters that deviate from estimates in Table 4 by no more than two standard posterior deviations. Therefore, our approach is closer to that of Drechsler (2013), who introduces an upper bound on the entropy of the worst-case scenario relative to the true model.

Under the choice of the penalty term as in Liu, Pan, and Wang (2005) pricing of assets is performed as if the SDF takes the same form as in (6), while the no-arbitrage condition holds under the worst-case scenario. Let us denote the probability measure that corresponds to this scenario by $\mathbb{P}^*$. Under $\mathbb{P}^*$, the dynamics of the market return are still described by equation (7), but the intensity of the Poisson process $N^R_t$ is higher $\lambda^*_{R,t} > \lambda_{R,t}$ and the distribution of jumps $\xi^R_{t}$ differs in an unfavorable way. Therefore, the no-arbitrage condition now states that $M_t R_t$ is a martingale under $\mathbb{P}^*$, where $M_t$ takes the same form as in (6) with a different value of $\gamma_0$. This result is obtained for the CRRA utility by Liu, Pan, and Wang (2005), but holds for other homogenous utility specifications.$^{21}$

$^{21}$To introduce the penalty term as in Liu, Pan, and Wang (2005) for the Epstein-Zin utility, we can define the discrete-time utility as follows: $U_t = c_1^{-\psi} h + e^{-\delta h} \left[ \mathbb{E}^{\mathbb{P}^*}_{U_t} (U_{t+h} + H(\mathbb{P}^*)U_{t+h}h)^{\theta} \right]^{\frac{1}{\theta}}$, where $H(\mathbb{P}^*)$ is a penalty for deviation of $\mathbb{P}^*$ from the true model. The continuous-time analog of this utility is obtained by taking the limit
Following the steps outlined in the appendix, we can calculate the total premium for the market return:

$$\pi_{R,t-} = \gamma_1 V_t^R + \gamma_2 \text{cov}_t^c(R_t, R_{\perp,t}) + \lambda_{R,t-} E_{t-} \left[ \xi_t^R \right] + \lambda_{R,t-}^* E_{t-}^* \left[ (1 + \xi_t^R)^{-\gamma} - (1 + \xi_t^R)^{1-\gamma} \right],$$

where the expectation $E_{t-}^*$ is constructed under $\mathbb{P}^*$. Similarly, for the factor returns we can derive:

$$\pi_{f,t-} = \pi_{+,t-} - \pi_{-,t-} = $$

$$\gamma_1 \text{cov}_t^c(F_t, R_t) + \gamma_2 \text{cov}_t^c(F_t, R_{\perp,t}) + \lambda_{R,t-} E_{t-} \xi_{i,t}^f - \lambda_{R,t-}^* E_{t-}^* \left[ \xi_{i,t}^f(1 + \xi_t^R)^{-\gamma} \right].$$

For our analysis, we allow $\lambda_{R,t}^*$ to deviate from $\lambda_{R,t}$ by two posterior standard deviations. We focus on the effect of uncertainty about jump arrivals alone and let $E_t = E_t^*$ in the above formulas. The next result applies the above formulas to the SVCJ model.

**Theorem 2.** Let the market return, momentum return, and value return follow the jump-diffusion dynamics given by system (2) with further restrictions imposed in Section 3. Suppose the stochastic discount factor is given by formula (6), where $R_t = F_{i,t}$ and $R_{\perp,t}$ is an index of unspecified additional factors with continuous diffusion dynamics. Suppose, the no-arbitrage condition holds under the alternative probability measure $\mathbb{P}^*$ which overestimates the intensity of market jumps $\lambda_1$, i.e., $\lambda_1^* > \lambda_1$. Then, the expected jump risk premium for the market returns equals

$$\lambda_1^* E \left[ \xi_{1,t}^f \right] + \lambda_1^* E \left[ (1 + \xi_{1,t}^f)^{-\gamma} - (1 + \xi_{1,t}^f)^{1-\gamma} \right],$$

where $\xi_{1,t}^f$ is conditionally normal $\mathcal{N}(\mu_{1,t}^f + \rho_{1}^f [\xi_{1,t}^v]^{\gamma},[\sigma_{1,t}^v]^{2})$ and $\xi_{1,t}^v$ is exponentially distributed $\exp(\mu_{1,t}^v)$. The expected jump risk premia for secondary factors ($i = 2, 3$) equal

$$\lambda_i^* E \left[ \xi_{i,t}^f \right] + \lambda_i^* E \left[ (1 + \xi_{i,t}^f)^{-\gamma} - (1 + \xi_{i,t}^f)^{1-\gamma} \right],$$

where $\xi_{i,t}^f$ is conditionally normal $\mathcal{N}(\mu_{i,t}^f + \rho_{i}^f [\xi_{i,t}^v]^{\gamma},[\sigma_{i,t}^v]^{2})$ and $\xi_{i,t}^v$ is exponentially distributed $\exp(\mu_{i,t}^v)$. $h \to 0$ and following the steps from Duffie and Epstein (1992) to obtain $U_t^0 = E_t^\mathbb{P}^* \left[ \int_{s=1}^{\infty} \left( \theta^* - \frac{1}{U_s} \right) U_s \right] ds$, where $\delta^* = \delta - H(\mathbb{P}^*)$. It follows that the pricing of assets under this utility occurs as if $\mathbb{P}^*$ were a true model and $\delta^*$ were the new time preference parameter. Note that $\delta^*$ only affects the scale of the stochastic discount factor, i.e., $\gamma_t$ in (6).
Figure 7 (dashed lines) plots the jump premia in the momentum and value, which now include the uncertainty premium as a function of the same for the market return. These lines are obtained by varying the loading $\gamma_1$ between 1.5 and 20. The uncertainty further decreases premium paid on the value factor and increases the jump premia in the market and momentum returns. The momentum factor is affected more because its average losses conditional upon jump are larger; see $E_\xi_{i,t}^f$ in Table 5. The same phenomenon is apparent in the portfolio selection exercise in Section 5 and manifests as the increased expected shortfall of momentum portfolios on jump days, see Table 7. Therefore, the new jump premium line is above the one that corresponds to the case without model uncertainty. The momentum jump premium is now approximately 40% to 60% of the market jump premium.

6.3 Model Uncertainty Premia for Individual Jumps

Our empirical results confirm that the momentum factor can jump in the absence of extreme market returns. Naturally, these individual jumps cannot be priced if they are, in fact, idiosyncratic. However, they will be priced if there is a simultaneous small jump in the market return that remains undetected. In this subsection, we evaluate a premium due to the model uncertainty about individual jumps $\tilde{N}_{i,t}$ if these jumps are viewed as negative by investors.

We can formally define this effect as follows. Suppose the market return includes a small jump component $\omega_{R,i}\xi_i^f dN_i^i$ related to the individual jump in the $i$th factor. The additional premium to the market return is small if $\omega_{R,i} > 0$ is small. However, the presence of this small component affects the worst-case scenario for the representative investor which now includes the deviation of the $\lambda_i^{*\ast}$ from $\lambda_i^i$. In fact, if $\omega_{R,i}$ is small, the additional premium to the $i$th factor does not depend on $\omega_{R,i}$ or $\gamma_1$ and is equal to

$$\lambda_{i,t} - E_{t-\xi_i^f} - \lambda_{i,t}^{*\ast} = E_{t-\xi_i^f}^{*\ast} - \xi_i^f.$$ 

Again, we choose $\lambda_{i,t}^{*\ast}$ that is two standard deviations above the estimate of $\lambda_i^i$ and assume that $E_{t-\xi_i^f} = E_{t-\xi_i^f}^{*\ast}$. For example, for the momentum factor, the estimate of $\lambda_{i,t}$ is 0.0019 with a standard deviation of 0.0010, which implies that, under the worst-case scenario, $\lambda_{i,t}^{*\ast}$ is 0.0039.

Figure 7 (dotted line) shows the total jump risk premia in the momentum and value premia that now includes the risk premia due to the co-jumps with the market return (solid line), premia
arising from the model uncertainty about market jumps arrival and the premia due to model uncertainty about the arrival of momentum and value jumps. The jump premium in the value factor is again negative. The total jump premium in the momentum return is now between 50% and 80% of the market jump premium. Hence, a significant part of the momentum premium can come from rare events. Note that our estimate is rather conservative, as we impose a one-factor structure on the jump risk in (6). Moreover, we note that a large part of the jump premium is due to the model uncertainty, while we ruled out some of it by assuming that the distribution of extreme returns is exactly known.

7 Conclusion

This paper proposes and estimates a multivariate jump-diffusion model for three fundamental stock market factors: the market return, value, and momentum. We analyze the co-interaction in extreme events between the factors. Our results strongly support the presence of co-jumps in the factor series. We observe that jumps in the market return greatly increase the odds of observing jumps in the other factors.

We provide a comprehensive analysis of the joint behavior of factors during extreme periods. We find that a combination of the market portfolio and the value factor reduces the expected loss, variance and VaR on days in which either factor jumps. We find similar results for adding the momentum portfolio to the market portfolio with the noted exception of an increase in the expected loss. Further, we analyze the effect of including an additional factor to a position in the market portfolio and a risk free bond for a risk averse investor, while considering the uncertainty over the factor’s future performance. We find that including the momentum factor dramatically decreases performance when volatility is high and these losses largely correspond to days on which jumps have non-negligible probabilities. We view this last finding as providing support for a relationship between extreme events and the momentum performance.

Finally, we apply our model to assess the magnitude of the value and momentum jump premia. We find that jumps could be a significant source of the momentum premium, but they are unlikely to contribute to the value premium.
A Proof of Theorem 1

Proof. We start by proving formula (8) for the market premium. We note that
\[
d\ln(M_t, R_t) = \gamma_0 dt - \gamma_2 d\ln R_{\perp,t} + (1 - \gamma_1) d\ln R_t,
\]
i.e., \( M_t R_t \) is a function of \( R_t \) and \( R_{\perp,t} \). We apply Itô’s lemma for jump-diffusion processes (see Cont and Tankov, 2004) and arrive at the following dynamics of the discounted returns \( M_t R_t \):

\[
\frac{dM_t R_t}{M_{t-} R_{t-}} = \left( \gamma_0 - \gamma_2 \left( \mu_{t-} - \frac{V_{t-}^\perp}{2} \right) + (1 - \gamma_1) \left( r_{t-} + \bar{\pi}_{R,t-} - \frac{V_{t-}^R}{2} \right) \right) dt
\]

\[
+ \frac{1}{2} \left( -\gamma_2 \ln R_{\perp} + (1 - \gamma_1) \ln R - \gamma_2 \ln R_{\perp} + (1 - \gamma_1) \ln R_t \right)
\]

\[
- \gamma_2 \sqrt{V_{t-}^\perp} dB_{t-}^i - (1 - \gamma_1) \sqrt{V_{t-}^R} dB_{t-}^R + dN_t^R ((1 + \xi_t^R)^{1-\gamma_1} - 1),
\]

where \( \mu_{t-} dt \) is the drift of the index \( \frac{dR_{\perp,t}}{R_{\perp,t}} \) and \( V_{t-}^\perp \) is its spot variance. We denote the continuous part of the quadratic variation (see definition 8.3 of Cont and Tankov, 2004) of the process \( X_t \) by \( [X, X]^c_t \). The entire quadratic variation term appearing above becomes a part of the drift of \( \frac{dM_t R_t}{M_{t-} R_{t-}} \).

By the no-arbitrage condition, the instantaneous increment to \( M_t R_t \), i.e. its drift, is conditionally mean zero. Therefore, we obtain that

\[
\left( \gamma_0 - \gamma_2 \left( \mu_{t-} - \frac{V_{t-}^\perp}{2} \right) + (1 - \gamma_1) \left( r_{t-} + \bar{\pi}_{R,t-} - \frac{V_{t-}^R}{2} \right) \right) dt
\]

\[
+ \frac{1}{2} \left( -\gamma_2 \ln R_{\perp} + (1 - \gamma_1) \ln R - \gamma_2 \ln R_{\perp} + (1 - \gamma_1) \ln R_t \right)
\]

\[
- \gamma_2 \sqrt{V_{t-}^\perp} dB_{t-}^i - (1 - \gamma_1) \sqrt{V_{t-}^R} dB_{t-}^R + dN_t^R ((1 + \xi_t^R)^{1-\gamma_1} - 1) \ dt = 0.
\]

Next consider a risk-free bond whose price \( B_t \) grows at the risk-free rate \( r_t \), \( dB_t/B_t = r_t dt \). Applying Itô’s lemma to \( M_t B_t \) as a function of processes \( R_t \), \( R_{\perp,t} \), and \( B_t \), we obtain the following dynamics of the discounted return on bond:

\[
\frac{dM_t B_t}{M_{t-} B_{t-}} = \left( \gamma_0 + r_{t-} - \gamma_2 \left( \mu_{t-} - \frac{V_{t-}^\perp}{2} \right) - \gamma_1 \left( r_{t-} + \bar{\pi}_{R,t-} - \frac{V_{t-}^R}{2} \right) \right) dt
\]

\[
+ \frac{1}{2} \left( -\gamma_2 \ln R_{\perp,t} - \gamma_1 \ln R_t - \gamma_2 \ln R_{\perp,t} - \gamma_1 \ln R_{t}^c \right)
\]

\[
- \gamma_2 \sqrt{V_{t-}^\perp} dB_{t-}^i - (1 - \gamma_1) \sqrt{V_{t-}^R} dB_{t-}^R + dN_t^R ((1 + \xi_t^R)^{\gamma_1} - 1) .
\]
Because the expected drift of the above process is zero, we have that

\[
\left( \gamma_0 + r_{t-} - \gamma_2 \left( \mu^R_{t-} - \frac{V^R_{t-}}{2} \right) - \gamma_1 \left( r_{t-} + \bar{\pi}_{R,t-} - \frac{V^R_{t-}}{2} \right) \right) dt \\
+ \frac{1}{2} d \left[ -\gamma_2 \ln R_{t,t} - \gamma_1 \ln R_t, -\gamma_2 \ln R_{t,t} - \gamma_1 \ln R_{i,t} \right]_t^c + \lambda_{R,t-} \mathbb{E}_{t-} \left( (1 + \xi^R_t)^{-\gamma_1} - 1 \right) dt = 0. \tag{11}
\]

Subtracting condition (11) from (10), we obtain that

\[
\pi_{R,t-} dt = \bar{\pi}_{R,t-} dt + \lambda^R_{R,t-} \mathbb{E}_{t-} \xi^R dt = \\
\gamma_1 V^R_{t-} dt + \gamma_2 d \left[ \ln R, \ln R_{t,t} \right]_t^c + \lambda_{R,t-} \mathbb{E}_{t-} \left[ \xi^R + (1 + \xi^R_t)^{-\gamma_1} - (1 + \xi^R_t)^{1-\gamma_1} \right] dt,
\]

where \( \ln R, \ln R_{t,t} \) denotes the continuous part of the quadratic covariance between processes \( \ln R_t \) and \( \ln R_{t,t} \). We use another notation in Section 6 for the instantaneous continuous covariance \( \text{cov}_{t-} (R_t, R_{t,t}) \) that denotes the ratio \( \frac{d \ln R_t \ln R_{t,t}}{dt} = \frac{d[R,R_{t,t}]}{R_t R_{t,t} dt} \frac{1}{dt} \).

We next prove formula (9) for other factors. We follow the same steps as for the market return and derive the dynamics of \( M_t R_{i,t} \):

\[
\frac{d M_t R_{i,t}}{M_t R_{i,t}} = \gamma_0 dt - \gamma_1 d \ln R_t - \gamma_2 d \ln R_{t,t} + d \ln R_{i,t} \\
= \left( \gamma_0 - \gamma_1 \left( r_{t-} + \bar{\pi}_{t-} - \frac{V^R_{t-}}{2} \right) - \gamma_2 \left( \mu^i_{t-} - \frac{V^i_{t-}}{2} \right) \right) dt \\
- \gamma_2 \sqrt{V^R_{t-} dB^R_t} - \gamma_1 \sqrt{V^R_t dB^R_t} + \sqrt{V^i_t dB^i_t} \\
+ \frac{1}{2} d \left[ -\gamma_2 \ln R_{t,t} - \gamma_1 \ln R_t + \ln R_{i,t,} - \gamma_2 \ln R_{t,t} - \gamma_1 \ln R_t + \ln R_{i,t} \right]_t^c \\
+ \left( (1 + \xi^R_t)^{-\gamma_1} (1 + \xi^c_t) - 1 \right) d N^R_t + \xi^c_t d N^i_t.
\]

Combining the no-arbitrage condition for return \( R_{i,t} \) with (11), we obtain that

\[
\pi_{i,t} dt = \gamma_1 d \left[ \ln R_t, R_{i,t} \right]_t^c + \gamma_2 d \left[ \ln R_{t,}, \ln R_{i,t} \right] + \lambda_{R,t-} \mathbb{E}_{t-} \left[ \xi^c - (1 + \xi^R_t)^{-\gamma_1} \xi^c \right] dt.
\]

Finally, the factor premium is the difference in premia paid on the long and short positions of the factor portfolio. Note that \( \pi^c_t \) is linear in the return \( R_{i,t} \) and cojump \( \xi^c_t \). The factor return is the difference in returns between its long and short positions. Likewise, the factor’s co-jump with the market return is the difference between the corresponding co-jumps. Therefore, the formula above also holds for the factors.
The final step of the proof is to apply the above formulas to the SVCJ model presented in this paper. First, the intensity of the market jumps is constant, \( \lambda^R_t = \lambda_1 \). Similarly, the distribution of market jumps \( \xi^R_t = \xi^f_{1,t} \) does not depend on time. Therefore, for the market return, we have that

\[
\lambda_{R,t-} \mathbb{E}_{t-} \left[ (1 + \xi^R_t)^{-\gamma_1} - (1 + \xi^R_{1,t})^{-\gamma_1} \right] = \lambda_1 \mathbb{E} \left[ (1 + \xi^f_{1,t})^{-\gamma_1} - (1 + \xi^f_{1,t})^{-\gamma_1} \right].
\]

For the other two factors, the cojump sizes are \( \xi^c_t = \beta_i^f \xi^f_{i,t} + \xi^f_{i,t}, i = 2, 3 \). Because the factor jump premia are linear in \( \xi^c_t \), we can write

\[
\lambda_{R,t-} \mathbb{E}_{t-} \left[ (1 + \xi^R_t)^{-\gamma_1} \xi^c_t \right] = \lambda_1 \beta_i^f \mathbb{E} \left[ (1 + \xi^f_{i,t})^{-\gamma_1} \xi^f_{i,t} \right] + \lambda_1 \mathbb{E} \left[ (1 + \xi^f_{i,t})^{-\gamma_1} \xi^f_{i,t} \right].
\]

Finally, note that \( \xi^f_{i,t} \) is zero with probability \( (1 - \lambda_{i,t}) \) and has the same distribution as \( \xi^R_t \), otherwise. Therefore,

\[
\lambda_{R,t-} \mathbb{E}_{t-} \left[ (1 + \xi^R_t)^{-\gamma_1} \xi^f_{i,t} \right] = \lambda_1 \beta_i^f \mathbb{E} \left[ (1 + \xi^f_{i,t})^{-\gamma_1} \xi^f_{i,t} \right] + \lambda_1 \lambda_{i,t} \mathbb{E} \left[ (1 + \xi^f_{i,t})^{-\gamma_1} \xi^f_{i,t} \right],
\]

\( \square \)

**B Proof of Theorem 2.**

*Proof.* Consider pricing of the market asset under the model uncertainty. Suppose, \( \lambda^*_R \) is the current intensity of jumps \( N^R_t \) under the worst-case scenario \( \mathbb{P}^* \). Following the same steps as in the proof of Theorem 1 for equations (10) and (11), we can show that the following pricing conditions hold under \( \mathbb{P}^* \):

\[
\left( \gamma_0 - \gamma_2 \left( \mu^1_{i,-} - \frac{V^1_{i,-}}{2} \right) \right) + (1 - \gamma_1) \left( r_{i,-} + \bar{\pi}_{R,t,-} - \frac{V^R_{i,-}}{2} \right) dt
\]

\[
+ \frac{1}{2} d \left[ (-\gamma_2 \ln R_{1,t} + (1 - \gamma_1) \ln R_{i,t} - \gamma_2 \ln R_{1,t} + (1 - \gamma_1) \ln R_{i,t}^c \right]
\]

\[
+ \lambda^*_R \mathbb{E}_{t-}^* \left( (1 + \xi^R_t)^{-\gamma_1} - 1 \right) dt = 0,
\]

and

\[
\left( \gamma_0 + r_{i,-} - \gamma_2 \left( \mu^1_{i,-} - \frac{V^1_{i,-}}{2} \right) - \gamma_1 \left( r_{i,-} + \bar{\pi}_{R,t,-} - \frac{V^R_{i,-}}{2} \right) \right) dt
\]

\[
+ \frac{1}{2} d \left[ (-\gamma_2 \ln R_{1,t} - \gamma_1 \ln R_{i,t} - \gamma_2 \ln R_{1,t} - \gamma_1 \ln R_{i,t}^c \right] + \lambda^*_R \mathbb{E}_{t-}^* \left( (1 + \xi^R_t)^{-\gamma_1} - 1 \right) dt = 0.
\]
Combining these two conditions we find the following expression for $\tilde{\pi}_{R,t-}$:

$$
\tilde{\pi}_{R,t-} dt = \gamma_1 V_t^R dt + \gamma_2 d[\ln R, \ln R_t] + \lambda_{R,t-}^* E_t^* \left[ \xi_t^R + (1 + \xi_t^R)^{-\gamma_1} - (1 + \xi_t^R)^{1-\gamma_1} \right] dt - \lambda_{R,t-}^* E_t^* \xi_t^R dt.
$$

Premium $\pi_{R,t-}$ is obtained by subtracting average jump losses,

$$
\pi_{R,t-} dt = \gamma_1 V_t^R dt + \gamma_2 d[\ln R, \ln R_t] + \lambda_{R,t-}^* E_t^* \left[ \xi_t^R + (1 + \xi_t^R)^{-\gamma_1} - (1 + \xi_t^R)^{1-\gamma_1} \right] dt - \lambda_{R,t-}^* E_t^* \xi_t^R dt + \lambda_{R,t}^* E_t \xi_t^R dt,
$$
or

$$
\pi_{R,t-} dt = \gamma_1 V_t^R dt + \gamma_2 d[\ln R, \ln R_t] + \lambda_{R,t-}^* E_t^* \left[ (1 + \xi_t^R)^{-\gamma_1} - (1 + \xi_t^R)^{1-\gamma_1} \right] dt + \lambda_{R,t}^* E_t \xi_t^R dt.
$$

The formulas for the premia for the other two factors are obtained analogously.
Table 1: Summary Statistics

This table reports summary statistics for the daily excess returns of the three stock market factor portfolios. The time period runs from January 1, 1980 through March 31, 2013. The daily returns are in percentage form.

<table>
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<th>Mean</th>
<th>Volatility</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min</th>
<th>Max</th>
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<td>-1.1957</td>
<td>17.7388</td>
<td>-8.6648</td>
<td>6.8126</td>
</tr>
</tbody>
</table>
Table 2: Univariate Model (SVCJ) Parameters

This table reports posterior means for the univariate SVCJ specification for daily excess returns of the three stock market factor portfolios. Parameter estimates correspond to percentage daily excess returns. The time period considered runs from January 1, 1980 through March 31, 2013. Posterior standard deviations are reported in parentheses. The SVCJ model for factor $F_t$ returns is defined as follows:

$$\frac{dF_t}{F_{t-}} = \mu \, dt + \sqrt{V_t} \, dB_t + \xi_{f_t} \, dN_t,$$

$$dV_t = \kappa (\theta - V_{t-}) dt + \sigma_v \sqrt{V_t} \, dBv_t + \xi_{v_t} \, dN_t,$$

where $\xi_{t}^v$ has density $f_{\xi_{t}^v}(x) = \mu_{t}^{v,J} \exp(-\mu_{t}^{v,J} x)$ for $x \geq 0$ and $\xi_{t}^f | \xi_{t}^v \sim N(\mu_{t}^J + \rho^J \xi_{t}^v, (\sigma^J)^2)$. $N_t$ is a Poisson process with intensity $\lambda$ and $B_t$ and $Bv_t$ are two Brownian motions with correlation $\rho$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MKT</th>
<th>HML</th>
<th>UMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.0409 (0.0084)</td>
<td>0.0069 (0.0040)</td>
<td>0.0522 (0.0047)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.0250 (0.0075)</td>
<td>0.0198 (0.0027)</td>
<td>0.0194 (0.0155)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.6398 (0.0704)</td>
<td>0.1261 (0.0132)</td>
<td>0.2022 (0.0364)</td>
</tr>
<tr>
<td>$\sigma_v^2$</td>
<td>0.0132 (0.0027)</td>
<td>0.0018 (0.0003)</td>
<td>0.0040 (0.0009)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.0079 (0.0025)</td>
<td>0.0046 (0.0012)</td>
<td>0.0058 (0.0032)</td>
</tr>
<tr>
<td>$\mu^J$</td>
<td>-1.8815 (0.8058)</td>
<td>0.4814 (0.5174)</td>
<td>-1.3342 (0.6980)</td>
</tr>
<tr>
<td>$\rho^J$</td>
<td>-0.5045 (0.6178)</td>
<td>0.2073 (0.6286)</td>
<td>-0.3685 (0.6255)</td>
</tr>
<tr>
<td>$(\sigma^J)^2$</td>
<td>3.6664 (1.2821)</td>
<td>1.9197 (0.5532)</td>
<td>2.8064 (0.9309)</td>
</tr>
<tr>
<td>$\mu^{v,J}$</td>
<td>0.8769 (0.2049)</td>
<td>1.5875 (0.2770)</td>
<td>1.0101 (0.2262)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.6188 (0.0362)</td>
<td>0.0367 (0.0490)</td>
<td>0.1855 (0.0466)</td>
</tr>
</tbody>
</table>
Table 3: Jump-Diffusion Variance Decomposition

This table reports the variance decomposition of daily returns (in %) into continuous and jump components for each of the three stock market factor portfolios. Posterior standard deviations for each component are given in parentheses. The continuous part of the variance is the average variance of returns in the absence of jumps, i.e., $\theta$ in the SVCJ model. The jump part is the difference between the total average variance and $\theta$.

<table>
<thead>
<tr>
<th></th>
<th>Continuous</th>
<th>Jump</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>MKT</td>
<td>0.6398</td>
<td>0.4518</td>
<td>1.0916</td>
</tr>
<tr>
<td></td>
<td>(0.0704)</td>
<td>(0.1005)</td>
<td>(0.0917)</td>
</tr>
<tr>
<td>HML</td>
<td>0.1261</td>
<td>0.1605</td>
<td>0.2865</td>
</tr>
<tr>
<td></td>
<td>(0.0132)</td>
<td>(0.0381)</td>
<td>(0.0366)</td>
</tr>
<tr>
<td>UMD</td>
<td>0.2022</td>
<td>0.3509</td>
<td>0.5531</td>
</tr>
<tr>
<td></td>
<td>(0.0364)</td>
<td>(0.0858)</td>
<td>(0.0790)</td>
</tr>
</tbody>
</table>
This table reports posterior means for the multivariate SVCJ specification for three factor returns (in %): market, momentum, and value. Posterior standard deviations are reported in parentheses. The time period considered runs from January 1, 1980 through March 31, 2013. The multivariate SVCJ model is defined as follows. For each factor $F_{i,t}$, $i = 1, 2, 3$:

$$
\frac{dF_{i,t}}{F_{i,t}} = \mu_i dt + \sqrt{V_{i,t}} dB_{i,t} + \xi^f_{i,t} dN_{i,t},
$$

$$
dV_{i,t} = \left( \sum_{j=1}^{3} \kappa_{i,j} (\theta_j - V_{j,t-}) \right) dt + \sigma_{i,v} \sqrt{V_{i,t}} dB_{v,t} + \xi^v_{i,t} dN_{i,t},
$$

where $\xi^f_{i,t} dN_{i,t} = \xi^f_{i,t} c dN_{1,t} + \tilde{\xi}^f_{i,t} d\tilde{N}_{i,t}$ and $\xi^v_{i,t} dN_{i,t} = \xi^v_{i,t} c dN_{1,t} + \tilde{\xi}^v_{i,t} d\tilde{N}_{i,t}$, $i = 2, 3$. Shocks $(dB_{1,t}, dB_{2,t}, dB_{3,t}, d\tilde{B}_{1,t}, d\tilde{B}_{2,t}, d\tilde{B}_{3,t})$ are Brownian with correlations indexed in this order, e.g., $\rho_{1,5}$ is the correlation between $dB_{1,t}$ and $d\tilde{B}_{2,t}$. Processes $N_{1,t}$, $\tilde{N}_{2,t}$, and $\tilde{N}_{3,t}$ are independent Poisson with intensities $\lambda_i$, $i = 1, 2, 3$. Random variables $\tilde{\xi}^f_{i,t}$ and $\tilde{\xi}^v_{i,t}$ have densities $f_i(x) = \mu_i^v \exp(-\mu_i^v x)$ for $x \geq 0$ and $\tilde{\xi}^f_{i,t} \equiv \xi^f_{i,t}$, $\tilde{\xi}^v_{i,t} \equiv \xi^v_{i,t}$, and $\tilde{\xi}^f_{i,t}$ are conditionally normal $N(\mu^f + \rho^f \tilde{\xi}^v_{i,t}, (\sigma^f)^2)$, $i = 1, 2, 3$. For momentum and value, $(\xi^f_{i,t}, \xi^v_{i,t})$, $i = 2, 3$ are zeros with probabilities $1 - \lambda_{i,c}$ and have the same distribution as $(\xi^f_{i,t}, \xi^v_{i,t})$, otherwise.

<table>
<thead>
<tr>
<th></th>
<th>MKT ($i = 1$)</th>
<th>UMD ($i = 2$)</th>
<th>HML ($i = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_i$</td>
<td>0.046 (0.008)</td>
<td>0.058 (0.010)</td>
<td>0.010 (0.004)</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>0.694 (0.033)</td>
<td>0.262 (0.021)</td>
<td>0.166 (0.008)</td>
</tr>
<tr>
<td>$\kappa_{i,1}$</td>
<td>0.048 (0.007)</td>
<td>0.005 (0.002)</td>
<td>0.004 (0.001)</td>
</tr>
<tr>
<td>$\kappa_{i,2}$</td>
<td>-0.050 (0.012)</td>
<td>0.002 (0.003)</td>
<td>-0.005 (0.002)</td>
</tr>
<tr>
<td>$\kappa_{i,3}$</td>
<td>0.019 (0.014)</td>
<td>0.002 (0.005)</td>
<td>0.012 (0.002)</td>
</tr>
<tr>
<td>$\rho_{1,i}$</td>
<td>0.100 (0.013)</td>
<td>-0.470 (0.010)</td>
<td></td>
</tr>
<tr>
<td>$\rho_{2,3}$</td>
<td>-0.114(0.013)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_{i,4}$</td>
<td>-0.469(0.010)</td>
<td>0.004 (0.031)</td>
<td>0.121(0.033)</td>
</tr>
<tr>
<td>$\rho_{i,5}$</td>
<td>-0.290(0.041)</td>
<td>0.161 (0.033)</td>
<td>0.106(0.032)</td>
</tr>
<tr>
<td>$\rho_{i,6}$</td>
<td>-0.359(0.043)</td>
<td>0.071 (0.035)</td>
<td>0.106(0.034)</td>
</tr>
<tr>
<td>$\rho_{i,4+3}$</td>
<td>0.887 (0.028)</td>
<td>0.876(0.022)</td>
<td></td>
</tr>
<tr>
<td>$\rho_{5,6}$</td>
<td>0.927(0.016)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma^v_i$</td>
<td>0.133 (0.007)</td>
<td>0.078 (0.004)</td>
<td>0.057(0.004)</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>0.0179 (0.0022)</td>
<td>0.0019 (0.0010)</td>
<td>0.0007(0.0005)</td>
</tr>
<tr>
<td>$\lambda_{i,c}$</td>
<td>0.962 (0.035)</td>
<td>0.844 (0.086)</td>
<td></td>
</tr>
<tr>
<td>$\mu^J_i$</td>
<td>1.000(0.383)</td>
<td>-0.470 (0.227)</td>
<td>-0.258 (0.231)</td>
</tr>
<tr>
<td>$\rho^J_i$</td>
<td>-4.694(0.898)</td>
<td>-4.760 (1.379)</td>
<td>6.138 (2.000)</td>
</tr>
<tr>
<td>$\sigma^J_i$</td>
<td>2.010(0.244)</td>
<td>1.222 (0.141)</td>
<td>1.208 (0.129)</td>
</tr>
<tr>
<td>$\mu^v_i$</td>
<td>2.695(0.368)</td>
<td>6.774 (0.867)</td>
<td>9.762 (1.732)</td>
</tr>
</tbody>
</table>
Table 5: **Interpretation of Parameters: Multivariate Model (SVCJ)**

This table reports posterior means for derived statistics based on SVCJ estimates for factor returns in Table 4. Parameter estimates correspond to the percentage daily returns. Posterior standard deviations are reported in parentheses. The time period considered runs from January 1, 1980 through March 31, 2013. In Panel A, the table reports (i) correlations of the momentum and value volatilities with the market volatility, $\rho_{mkt.vol}$, (ii) autocorrelations of the slow and fast volatility components obtained as linear combinations of the factor volatilities, and (iii) variance decomposition of the factor volatilities into the slow and fast components. In Panel B, the table reports (i) correlations of the factors with the slow volatility component, (ii) the partial correlation with the fast volatility component, (iii) the total variation of the factor returns explained by correlations with volatilities, and (iv) partial correlations with the market return controlling for the effect of volatilities. In Panel C, the table reports expected losses in factor returns conditional upon jump.

<table>
<thead>
<tr>
<th></th>
<th>MKT</th>
<th>UMD</th>
<th>HML</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A: Vol Interrelations</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_{mkt.vol}$</td>
<td>0.887 (0.028)</td>
<td>0.876 (0.022)</td>
<td></td>
</tr>
<tr>
<td>Autocorr. slow vol</td>
<td>0.990 (0.002)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Autocorr. fast vol</td>
<td>0.966 (0.004)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Var. decomp. (slow)</td>
<td>0.245 (0.097)</td>
<td>0.827 (0.059)</td>
<td>0.789 (0.088)</td>
</tr>
<tr>
<td>Var. decomp. (fast)</td>
<td>0.622 (0.102)</td>
<td>0.085 (0.048)</td>
<td>0.113 (0.072)</td>
</tr>
<tr>
<td><strong>B: Cont. Leverage</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>... wrt slow vol</td>
<td>-0.291 (0.041)</td>
<td>0.161 (0.033)</td>
<td>0.106 (0.032)</td>
</tr>
<tr>
<td>... wrt fast vol(partial)</td>
<td>-0.453 (0.065)</td>
<td>-0.305 (0.060)</td>
<td>0.057 (0.062)</td>
</tr>
<tr>
<td>... total leverage $R^2$</td>
<td>0.317 (0.060)</td>
<td>0.145 (0.048)</td>
<td>0.024 (0.010)</td>
</tr>
<tr>
<td>residual factor corr. with MKT</td>
<td>-0.009 (0.048)</td>
<td>-0.495 (0.030)</td>
<td></td>
</tr>
<tr>
<td><strong>C: Jump Leverage</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>... $E_{t,t}^{\xi_f}$</td>
<td>-0.753 (0.299)</td>
<td>-1.177 (0.176)</td>
<td>0.375 (0.160)</td>
</tr>
</tbody>
</table>
Table 6: Certainty Equivalents for Factor Portfolios

The table reports the difference in certainty equivalents between the portfolio that includes only the market return and risk-free bond and the two portfolios that also include momentum (UMD) or value (HML) factors. The certainty equivalents are constructed for an investor who forms his portfolios daily and maximizes \((1 - \gamma)^{-1} \mathbb{E}_t(W_t R_{t,t+1})^{1-\gamma}\) at each \(t\), where \(W_t\) is his wealth at the end of day \(t\) and \(R_{t,t+1}\) is the daily total return. The conditional expectation is constructed under the SVCJ model, in which average returns follow a random walk path and are filtered based on the prior data. The certainty equivalents are calculated in annualized percentage units. The standard errors are reported in parentheses. The time period considered runs from January 1, 1980 through March 31, 2013.

<table>
<thead>
<tr>
<th>(\gamma)</th>
<th>CE(RF-MKT-UMD) - CE(RF-MKT)</th>
<th>CE(RF-MKT-HML) - CE(RF-MKT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma = 2)</td>
<td>10.9% (5.8%)</td>
<td>13.4% (2.9%)</td>
</tr>
<tr>
<td>(\gamma = 5)</td>
<td>8.5% (4.5%)</td>
<td>12.4% (2.4%)</td>
</tr>
<tr>
<td>(\gamma = 10)</td>
<td>6.9% (2.8%)</td>
<td>9.4% (2.0%)</td>
</tr>
<tr>
<td>(\gamma = 20)</td>
<td>3.7% (1.5%)</td>
<td>6.0% (1.4%)</td>
</tr>
</tbody>
</table>
Table 7: **Factor Portfolios: Performance on Jump Days**

This table reports mean returns, variances, and the Value-at-Risk (VaR) on jump days for factor portfolios. Factor portfolios are daily re-balanced and include $w_1$ share of the market return and $w_2$ share of the momentum return (left columns) or value return (right columns). The entries are estimated based on the parameters from Table 4. Frames indicate entries that are statistically equal at 5% significance level. Arrows with asterisks indicate directions in which the changes in entries are statistically significant.

<table>
<thead>
<tr>
<th>$w_2$</th>
<th>MKT-UMD</th>
<th>MKT-HML</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>Variance</td>
<td>VaR</td>
</tr>
<tr>
<td>-------</td>
<td>----------</td>
<td>---------</td>
</tr>
<tr>
<td>0.0</td>
<td>-0.68%</td>
<td>3.75 ↓*</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.73%</td>
<td>3.05 ↓*</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.77%</td>
<td>2.46 ↓*</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.82%</td>
<td>1.97 ↓*</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.86%</td>
<td>1.59 ↓*</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.91%</td>
<td>1.31 ↓*</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.95%</td>
<td>1.14</td>
</tr>
<tr>
<td>0.7</td>
<td>-1.00%</td>
<td>1.08</td>
</tr>
<tr>
<td>0.8</td>
<td>-1.05%</td>
<td>1.11</td>
</tr>
<tr>
<td>0.9</td>
<td>-1.09%</td>
<td>1.26 ↑*</td>
</tr>
<tr>
<td>1.0</td>
<td>-1.14%</td>
<td>1.51 ↑*</td>
</tr>
</tbody>
</table>
Figure 1: **Daily Returns.** The figure shows daily returns in % form for each of the three stock market factors.
Figure 2: **Daily Spot Volatilities and Volatility Jumps.** This figure shows filtered daily spot variances (\(V_t\)) and corresponding volatility jumps for each of the three stock market factors.
Figure 3: **Estimated Jump Probabilities.** This figure shows the time series of the estimated jump probabilities, $\mathbb{P}(J_t = 1)$, for each of the three stock market factors.
Figure 4: **Jump Probabilities and Z-statistics.** The figure shows the Z-statistics, \((F_t/F_{t-1} - 1)/\sqrt{V_{t-1}}\) with marks that correspond to days with estimated jump probabilities \(P(J_t = 1)\) exceeding 20% for each of the three stock market factors.
Figure 5: **Time Variation in Factor Premia.** The figure shows the estimated dynamics of the daily (in %) factor premia. The premia are modeled by random walks and are estimated jointly with the other parameters of the *SVCJ* model. Gray lines correspond to premia estimated using the entire sample (smoothers). Black lines correspond to premia estimated using prior information (filters).
Figure 6: Performance of Factor Strategies and Volatility. The figure shows annualized certainty equivalents (black lines) for factor strategies calculated only on days with estimated market volatility $V_{1,t}$ above $V_1$. The thresholds $V_1$ are indicated on x-axis. Gray areas correspond to 90% confidence intervals calculated using block bootstrap with blocks of two years. Dashed lines correspond to certainty equivalents calculated only on days with low probabilities of jumps $P(J = 1) < 20\%$. The certainty equivalents are constructed for an investor who forms his portfolios daily and maximizes $(1 - \gamma)^{-1}E_t(W_tR_{t,t+1})^{1-\gamma}$, $\gamma = 2$, at each $t$, where $W_t$ is his wealth at the end of day $t$ and $R_{t,t+1}$ is the daily total return. The conditional expectation is constructed under the SVCJ model, in which average returns follow a random walk path and are filtered based on the prior data.
Figure 7: **Jump Premia in Momentum and Value Factors.** The figure shows the annualized (in %) jump risk premia in momentum and value factors relative to the jump risk premium in market returns. Premia are constructed based on the estimated SVCJ model and one-factor structure of the jump risk. In particular, the stochastic discount factor (SDF) is $d \ln M_t = -\gamma_0 - \gamma_1 \ln R_t - \gamma_2 d \ln R_{\perp,t}$, where $R_t$ is the market return and the other component $R_{\perp,t}$ has continuous dynamics. The dashed lines account for the model uncertainty about the intensity of the market jumps. The dotted lines account for the model uncertainty about the intensity of market, momentum, and value jumps. In both cases, it is assumed that model uncertainty moves investors to maximize their worst-case utility.
References


