Conservation Equations in Derivative Form

Nomenclature

Symbols	**Explanation**
B	fluid property of the finite sized blob and finite sized control volume
$\mathrm{D}B/\mathrm{D}t$	$\mathrm{d}B/\mathrm{d}t$ as we travel along with the fluid blob (Lagrangian frame of reference)
$\mathrm{d}B$	fluid property of an infinitesimal sized blob and differential control volume
$\mathrm{d}m$	differential mass
b	$\mathrm{d}B/\mathrm{d}m$
$\mathrm{d}\forall$	differential control volume (CV)
$\mathrm{d}\forall_b$	volume of the infinitesimal sized blob
$\vec{V}$	fluid velocity vector with components u, v and w in Cartesian space
∇	the gradient (a vector operator)
$\nabla \cdot \vec{V}$	the divergence of the fluid velocity vector ($\mathrm{div}(\vec{V})$)
$\mathrm{d}\vec{F}$	force vector acting on a differential mass
$\mathrm{d}Q$	heat added to a differential mass
$\mathrm{d}W$	work done by a differential mass

The Material Derivative

Recall that we derived a form of the Reynolds Transport Theorem (RTT) for a differential control volume (i.e., a point in space),

$$\frac{1}{\mathrm{d}\forall}\frac{\mathrm{D}\,(\mathrm{d}B)}{\mathrm{D}t} = \frac{\partial(b\rho)}{\partial t} + \nabla \cdot \left(b\rho\vec{V}\right) \ .$$

Alternately, we could have started with the original RTT,

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathrm{blob}} \mathrm{d}B = \frac{\partial}{\partial t}\int_{\mathrm{cv}} \mathrm{d}B + \int_{\mathrm{cs}} b\rho\left(\vec{V}\cdot\vec{n}\right)\mathrm{dA}$$

and if we assume that B, ρ and $\vec{V}$ are continuous throughout the blob and the control volume then we can use the divergence theorem from vector calculus

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathrm{blob}} \mathrm{d}B = \frac{\partial}{\partial t}\int_{\mathrm{cv}} \mathrm{d}B + \int_{\mathrm{cv}} \nabla\cdot\left(b\rho\vec{V}\right)\ \mathrm{d}\forall$$

or

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathrm{blob}} b\rho\ \mathrm{d}\forall_b = \frac{\partial}{\partial t}\int_{\mathrm{cv}} b\rho\ \mathrm{d}\forall + \int_{\mathrm{cs}} \nabla\cdot\left(b\rho\vec{V}\right)\ \mathrm{d}\forall \ .$$

If we consider the CV to be rigid then

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\text{blob}} b\rho \,\mathrm{d}\forall_b = \int_{\text{cv}} \left[\frac{\partial(b\rho)}{\partial t} + \nabla\cdot\left(b\rho\vec{V}\right)\right]\,\mathrm{d}\forall$$

and if we are only interested in the differential blob and differential control volume,

$$\boxed{\frac{\mathrm{D}\,(\mathrm{d}B)}{\mathrm{D}t} = \frac{\partial(\mathrm{d}B)}{\partial t} + \nabla\cdot\left(\vec{V}\,\mathrm{d}B\right)\ . \qquad (1)}$$

This is the differential form of the RTT which is also the definition of the material derivative evaluated at a point in space.

Please note that Eq. (1) is not a conservation equation but the translation between the change in some property of a fluid particle (differential fluid blob) as we travel along with it (Lagrangian frame of reference) and how that same property changes at some point in space (differential control volume) that we are observing (Eulerian frame of reference). We have to develop this translation between the Lagrangian and Eulerian frames of references because the conservations of mass, momentum and energy were originally derived for a Lagrangian frame of reference (PHY 101, MECH 211 and ENGI 200).

The Conservation of Mass

As previously mentioned, the conservation equations were originally derived in a Lagrangian frame of reference, so the conservation of mass for our fluid particle is written as

$$\frac{\mathrm{D}\,(\mathrm{d}m)}{\mathrm{D}t} = 0\ .$$

We can now use our translator to rewrite the conservation equation in a form we can used in fluid mechanics,

$$\frac{\mathrm{D}\,(\mathrm{d}m)}{\mathrm{D}t} = \frac{\partial(\rho\,\mathrm{d}\forall)}{\partial t} + \nabla\cdot\left(\rho\vec{V}\,\mathrm{d}\forall\right) = 0\ .$$

But since $\mathrm{d}\forall$ is rigid then the equation can be written as

$$\frac{1}{\mathrm{d}\forall}\frac{\mathrm{D}\,(\mathrm{d}m)}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right) = 0\ . \tag{2}$$

The specific forms of Eq. (4) were mentioned in class.

- Steady flow ($\rho = f(x,y,z)$):

  $$\nabla\cdot\left(\rho\vec{V}\right) = 0\ . \tag{3}$$

- Incompressible flow ($\rho =$ constant):

  $$\nabla\cdot\vec{V} = 0\ . \tag{4}$$

Material Derivative for b

We know from our work with the integral forms of the conservation equations that we're going to have to deal with the material derivative of $\mathrm{d}B$. Since we know mass is conserved, we might as well investigate the form of the material derivative of b, i.e.,

$$\frac{\mathrm{D}\left(\mathrm{d}B\right)}{\mathrm{D}t} = \frac{\mathrm{D}\left(b\ \mathrm{d}m\right)}{\mathrm{D}t} = \mathrm{d}m\frac{\mathrm{D}b}{\mathrm{D}t} + b\frac{\mathrm{D}\left(\mathrm{d}m\right)}{\mathrm{D}t} = \mathrm{d}m\frac{\mathrm{D}b}{\mathrm{D}t}\ .$$

From Eq. (1) we can write,

$$\frac{\mathrm{D}\left(\mathrm{d}B\right)}{\mathrm{D}t} = \mathrm{d}m\frac{\mathrm{D}b}{\mathrm{D}t} = \frac{\partial(b\ \mathrm{d}m)}{\partial t} + \nabla\cdot\left(b\ \mathrm{d}m\vec{V}\right)\ . \tag{5}$$

The application of the chain rule to the partial with respect to time is straightforward, but we better look at the divergence term a bit closer. In a Cartesian coordinate system $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. For the scalar ξ and the vector $\vec{V}$ we have,

$$\nabla\xi = \frac{\partial\xi}{\partial x}\vec{i} + \frac{\partial\xi}{\partial y}\vec{j} + \frac{\partial\xi}{\partial z}\vec{k} \quad \text{and} \quad \nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\ .$$

So with a second scalar c we can write,

$$\begin{aligned}\nabla\cdot\left(\xi c\vec{V}\right) &= \frac{\partial\left(\xi c u\right)}{\partial x} + \frac{\partial\left(\xi c v\right)}{\partial y} + \frac{\partial\left(\xi c w\right)}{\partial z} = \\ \xi c\frac{\partial\left(u\right)}{\partial x} + \xi c\frac{\partial\left(v\right)}{\partial y} + \xi c\frac{\partial\left(w\right)}{\partial z} &+ u\frac{\partial\left(\xi c\right)}{\partial x} + v\frac{\partial\left(\xi c\right)}{\partial y} + w\frac{\partial\left(\xi c\right)}{\partial z} = \\ &\xi c\nabla\cdot\vec{V} + \left(\vec{V}\cdot\nabla\right)\left(\xi c\right)\ .\end{aligned}$$

Returning to our problem,

$$\nabla\cdot\left(b\ \mathrm{d}m\vec{V}\right) = b\ \mathrm{d}m\nabla\cdot\vec{V} + b\left(\vec{V}\cdot\nabla\right)\mathrm{d}m + \mathrm{d}m\left(\vec{V}\cdot\nabla\right)b\ ,$$

and we can write

$$\frac{\mathrm{D}\left(\mathrm{d}B\right)}{\mathrm{D}t} = \mathrm{d}m\frac{\mathrm{D}b}{\mathrm{D}t} = b\frac{\partial(\mathrm{d}m)}{\partial t} + \mathrm{d}m\frac{\partial b}{\partial t} + b\ \mathrm{d}m\nabla\cdot\vec{V} + b\left(\vec{V}\cdot\nabla\right)\mathrm{d}m + \mathrm{d}m\left(\vec{V}\cdot\nabla\right)b\ .$$

or, bringing back the material derivative of $\mathrm{d}m$,

$$\begin{aligned}\mathrm{d}m\left[\frac{\mathrm{D}b}{\mathrm{D}t} - \left(\frac{\partial b}{\partial t} + \left(\vec{V}\cdot\nabla\right)b\right)\right] &= -b\left[\frac{\mathrm{D}(\mathrm{d}m)}{\mathrm{D}t} - \left(\frac{\partial(\mathrm{d}m)}{\partial t} + \mathrm{d}m\nabla\cdot\vec{V} + \left(\vec{V}\cdot\nabla\right)\mathrm{d}m\right)\right] = \\ &-b\left[\frac{\mathrm{D}(\mathrm{d}m)}{\mathrm{D}t} - \left(\frac{\partial(\mathrm{d}m)}{\partial t} + \nabla\cdot\left(\mathrm{d}m\vec{V}\right)\right)\right]\ .\end{aligned}$$

From the conservation of mass equation, we know that the right hand side of the equation above is zero. Therefore,

$$\boxed{\frac{\mathrm{D}b}{\mathrm{D}t} = \frac{\partial b}{\partial t} + \left(\vec{V} \cdot \nabla\right) b \ . \qquad (6)}$$

Note that because we made use of the conservation of mass, Eq. (6) applies *only* to b and *not* $\mathrm{d}B$. So Eq. (5) can be written as,

$$\rho\frac{\mathrm{D}b}{\mathrm{D}t} = \rho\frac{\partial b}{\partial t} + \rho\left(\vec{V} \cdot \nabla\right) b \ . \tag{7}$$

Testing Eq. (7) for the conservation of mass, we have $b = 1$ and so the equation is satisfied identically.

Conservation of Linear Momentum

The conservation of linear momentum is a statement of the fact that the time rate of change of the momentum of a mass that we're following is equivalent to the sum of the forces acting on that mass. Written in our fluid mechanics nomenclature for the differential mass, it would appear as,

$$\frac{\mathrm{D}\left(\vec{V}\mathrm{d}m\right)}{\mathrm{D}t} = \sum \mathrm{d}\vec{F} = \mathrm{d}m\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\ \mathrm{d}\forall\frac{\mathrm{D}\vec{V}}{\mathrm{D}t}$$

or

$$\rho\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \sum\frac{\mathrm{d}\vec{F}}{\mathrm{d}\forall} = \rho\left[\frac{\partial\vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla\right)\vec{V}\right] \ ,$$

since $b = \vec{V}$.

Conservation of Energy

The conservation of energy is a statement of the fact that the time rate of change of the total energy of a mass that we're following is equivalent to the sum of the rate of heat added to, and rate of work done *by*, the mass. For the differential mass we can write,

$$\frac{\mathrm{D}\left(e\ \mathrm{d}m\right)}{\mathrm{D}t} = \rho\ \mathrm{d}\forall\frac{\mathrm{D}e}{\mathrm{D}t} = \frac{\mathrm{D}\mathrm{d}Q}{\mathrm{D}t} - \frac{\mathrm{D}\mathrm{d}W}{\mathrm{D}t} \ ,$$

or

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} = \frac{1}{\mathrm{d}\forall}\frac{\mathrm{D}\left(\mathrm{d}Q\right)}{\mathrm{D}t} - \frac{1}{\mathrm{d}\forall}\frac{\mathrm{D}\left(\mathrm{d}W\right)}{\mathrm{D}t} \ ,$$

since $b = e$.