Critique of the von Mises Yield Criterion

Introduction

Commercial releases for finite element methods for stress analysis began in the early 1960's. At that time the main engineering focus was on materials that were ductile metals with equal yield stresses in tension and compression. The tension yielding (failure) of such a material was usually checked against three "Failure Criteria": the maximum principal tension stress (P1 in SolidWorks simulation), the maximum shear stress (INT), and the "Distortional Energy Theory" (VON) at a point.

The onset of material yielding in an axial tension test as predicted by the Distortional Energy Theory can be reduced to equating the test yield stress to an equivalent stress. That equivalent stress is known as the von Mises Stress [von Mises 1914]. It is NOT a component of the stress tensor, or one of the principal stresses, but it has the units of stress. That criterion for the onset of yielding due to distortional energy level is defined as

$$\sigma_M = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{yy}-\sigma_{xx}\right)^2 + (\sigma_{zz}-\sigma_{xx})^2 + \left(\sigma_{zz}-\sigma_{yy}\right)^2 + 6\left(\tau_{xy}{}^2 + \tau_{xz}{}^2 + \tau_{yz}{}^2\right)} = \sigma_{Yield\ Test}$$

Or

$$\sigma_M = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2} = \sigma_{Yield\ Test}$$

and at any point where $\sigma_M > \sigma_{Yield\ Test}$ the material is considered to have failed (yielded).

Due to that early interest in ductile metals and the large number of stress items than can be plotted the von Mises "stress" became the default stress plot. Any other stress measures of importance have to be specifically requested based on the user's knowledge of the material and the problem being simulated. For example, in SolidWorks Simulation the von Mises "stress" plot is the default of twelve important stress measures as noted below (right click **Results → Define Stress Plot**).

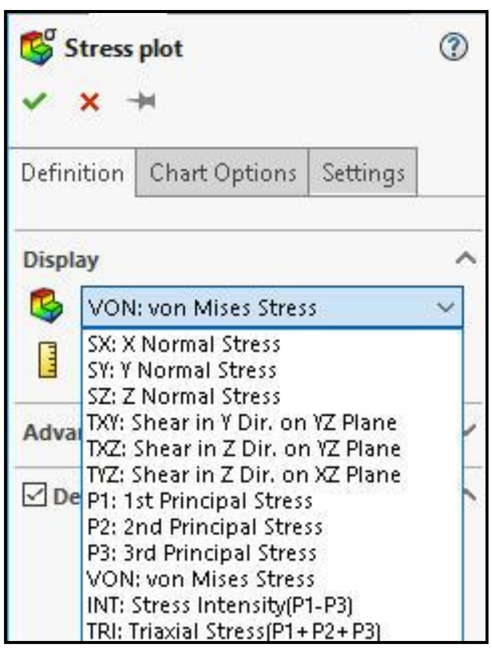

Figure 1 Default list of stress plots and graphs

The von Mises criterion is NOT a valid yield (failure) criterion for materials with different yield stress values in tension and compression. There are thousands of materials with a compressive yield stress different from their tensile yield stress. The compressive and tensile yield stresses for many such materials are available on material property data web sites, such as www.matweb.com. For example, cast iron has a compressive yield stress value that is two to four times its tensile yield stress value and thus a von Mises criterion is NOT valid for cast iron. But the von Mises "stress" appears as the default stress result, even for cast iron. Each engineer must have enough knowledge of the material being used in a simulation to properly decide which yield criteria need to be considered. As another example, human bone has a compressive yield stress value that is typically 70% higher than its tensile yield stress value. Yet again, most of the published studies on stresses in bone present the inappropriate von Mises values.

For a state of plane-stress ($\sigma_2 = 0$) the von Mises distortional energy measure 'yield curve' plots as an ellipse in the principle stress space,. Points inside the ellipse have not failed, points on the ellipse (yield curve) are impending to fail, and any stress points outside the ellipse denote failed (yielded) material points. The von Mises criterion agrees well with the test data for ductile metals, but not brittle materials

Stresses Review:

Finite element formulations often give the stress state at a point in terms of the Voigt stress vector notation as a condensed form of the symmetric stress tensor $[\boldsymbol{\sigma}]$:

$$\boldsymbol{\sigma}^T = [\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}] \Leftrightarrow [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}.$$

The state of stress at a point is completely defined by three normal and three shear stress components in reference to the global coordinate system XYZ. In general, the values of the stress components change if the coordinate system is rotated. At a certain orientation (X'Y'Z'), all shear stresses vanish and the state of stresses is completely defined by three normal stress components (P1, P2, P3). Those three normal stress components are referred to as "principal stresses" and the corresponding reference axes (X'Y'Z') are referred to as principal axes. The principal stresses are the eigenvalues of the stress tensor and the principal axes are their corresponding eigenvectors.

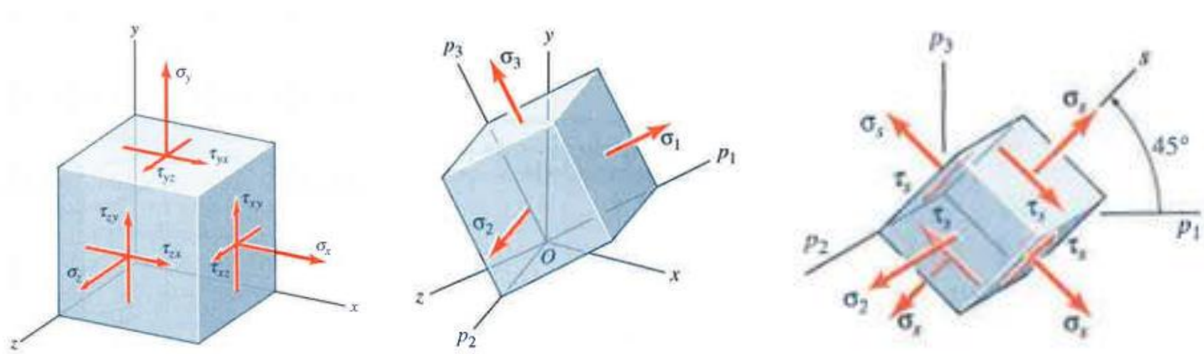

Figure 2 Stress tensor, Principal normal stresses, and Principal shear stresses

The three maximum shear stresses occur on three mutually perpendicular planes that are offset from the three planes of principal stress components by 45 degrees each. The largest of those three shear stresses is the maximum shear stress at the point, τ_{max}. It can be shown that in a axial tension test of a planar specimen at impending yield $\tau_{max} = \sigma_{Yield\ Test}/2$. So FEM codes allow plots of that value. However, other codes, including SolidWorks prefer to only compare to the yield data so it offers the intensity (INT $\equiv 2\,\tau_{max}$) as the value to compare to the yield stress.

The principal stresses also define other physically important quantities that relate to failure criteria, including the hydrostatic pressure (which is a third of the sum of the stress tensor diagonal values):

$$p = (\sigma_1 + \sigma_2 + \sigma_3)/3 = (\sigma_{xx} + \sigma_{yy} + \sigma_{33})/3 = \ 3\,TRI$$

The principal stresses, along with the hydrostatic pressure, are utilized to define several common material yield (failure) criteria theories, as will be discussed below.

Material with Multiple Yield Stress Values:

When a material yield condition is defined by more than one test value then a new notation is needed. Here, the absolute values of the experimentally measured yield stress values in tension, compression, and shear are denoted as $k_t, k_c,$ and k_s, respectively. In theory, it can take as many as 21 different tests to define the yield conditions of the most general material. However, in practice most common materials can be defined by one, or two, or three of the above yield test values. The ratio of the tension to compression yield stresses, $\kappa \equiv k_c/k_t$, is often referred to as the 'strength difference', and when that ratio is not unity the material is sometimes called an "asymmetric material". In theory, the von Mises criterion is only valid when $\kappa = 1$, but experience shows it still to be useful for values 'reasonably' near unity, say 1.3.

In the later sections a few alternate 'effective stress' criteria will be presented for materials that have different compressive and tensile yield stress test values, $\kappa \neq 1$. But how can that be done in SolidWorks (or other FEA codes) if such a criterion is not given in the SolidWorks stress options given above? Beginning with release 2017 of SolidWorks Simulation a user can define additional stress measures to be plotted by using a feature called the "Results Equation Editor". In SolidWorks Simulation select **Results → Results Equation Plot → Definition → Edit Equation**. SolidWorks provides assistance in creating any custom stress plot through the editor's pull-down menus for selecting the desired stress items, and the functions for combining them, as well as the usual arithmetic operations. In the results editor the full length stress names such as "TRI: Triaxial Stress (P1+P2+P3)", or "VON: von Mises Stress" can be selected from pull-down lists shown below.

When only the compressive and tensile yield stress values are required to define the yield conditions of a material then the three most commonly used 'equivalent stresses' are defined by the Burzynski criterion, the Hoffman criterion, and the Drucker-Prandtl criterion which will be discussed in the following sections. The user's knowledge of the material should serve as the guide for selecting one or all of the criteria to be displayed.

Burzynski criterion:

The Burzynski failure criterion [Burzynski 1929 2008, Fras 2010 2013 2014] applies to materials that have different yield stresses in tension and in compression. It states that a material yields when the volume change energy at a point equals the deviatoric energy in a tension test at the limit of elastic behavior. It can also be extended to include the shear failure value, k_s, as a third property. The Burzynski criterion can be calculated by combining the hydrostatic pressure and the von Mises stress along with the ratio of compressive and tensile yield stresses.

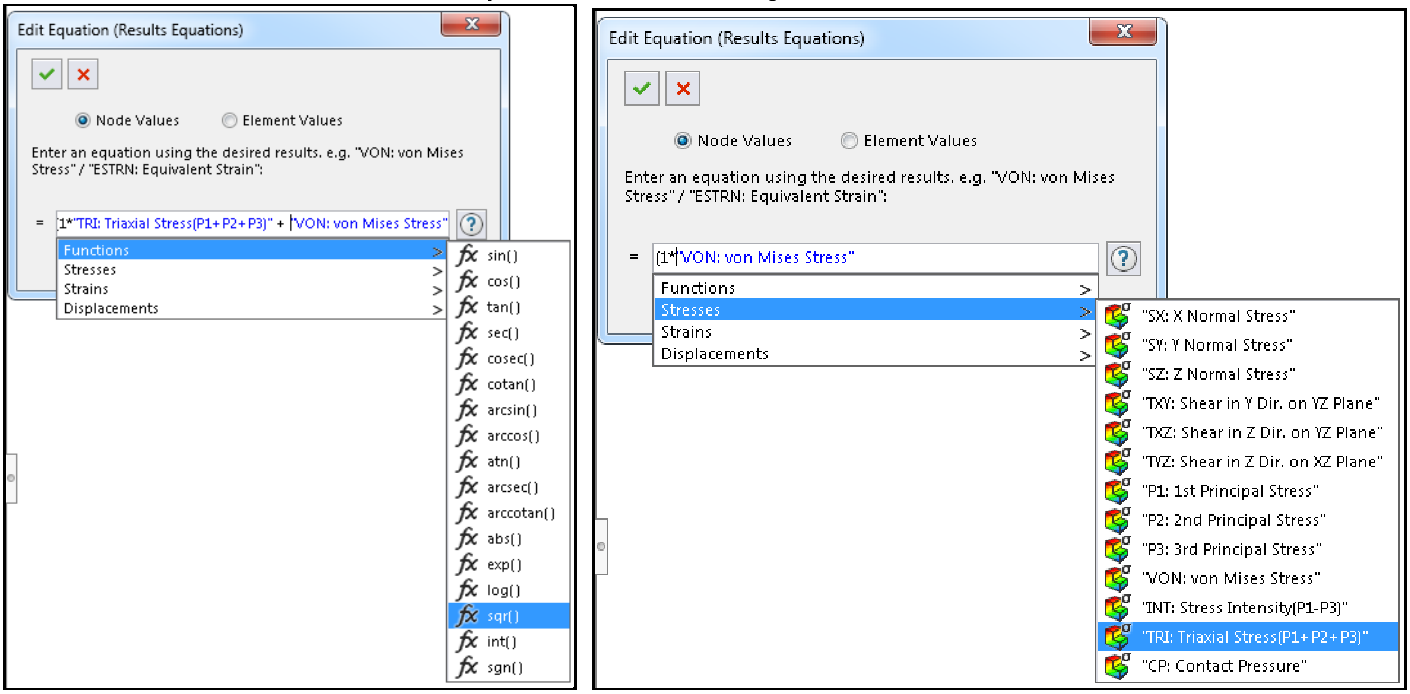

Figure 3 Available functions and stress quantities available in the SWS Results Editor

For a state of plane-stress $(\sigma_2 = 0)$ the Burzynski deviatoric energy measure also plots as an ellipse in the principle stress space. Figure 4 shows a Burzynski ellipse with the circular points representing the most common yield stress tests of: pure tension, pure compression, pure shear, bi-axial tension, and bi-axial compression. Note that the center of the Burzynski ellipse does not occur at the origin of the principle stress space (unless $k_t \equiv k_c$). When the compressive and tensile yield stresses differ, $k_t \neq k_c$, the Burzynski ellipse better fits yield or failure data points than the von Mises ellipse. Figure 5 shows a family of Burzynski ellipses with $k_c/k_t \equiv \kappa > 1$ compared to the dashed von Mises ellipse which requires $k_t \equiv k_c$. In Fig. 5, the single black point on the σ_1 axis is the tension failure stress, k_t, the black points on the negative σ_3 axis are different compression failure stress values, -k_c. The diagonal black points are alternate shear failure stress values, $\pm k_s$.

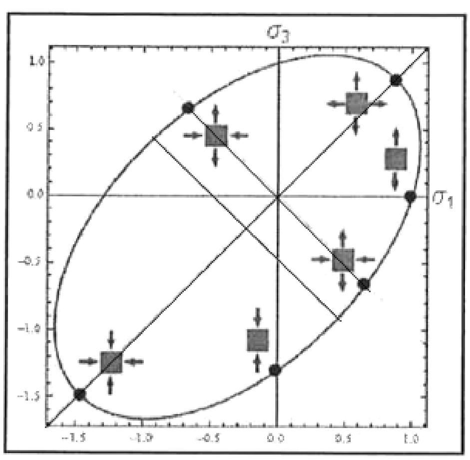

Figure 4 Yield stress on Burzynski ellipse

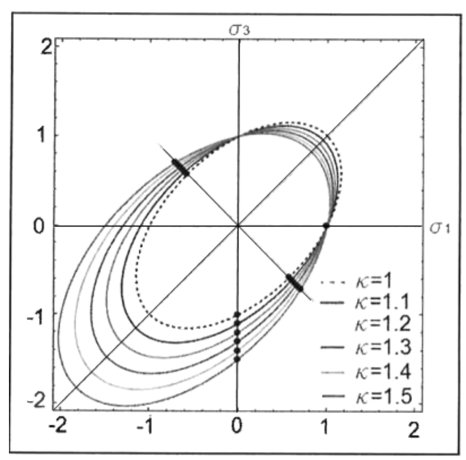

Figure 5 von Mises (dashed) and Burzynski ellipses

The dissertation of Fras [2013] includes detailed mathematical investigations of the Burzynski criterion. It also compared the criterion to published experimental data for about 25 materials with different tensile and compressive yield stresses. Those data were plotted in the principal stress space along with the Burzynski and the von Mises ellipses. In all cases the Burzynski criterion matched the yield data much better than the von Mises criterion.

The Burzynski criterion requires at least the k_c and k_t experimental yield stress measures. Equating the Burzynski measure in both tension and compression yield tests predicts impending material failure when

$$\sigma_B{}^2 = k_t k_c = k_t(\kappa\, k_t) = \kappa\, k_t^2.$$

That relation can be simplified to compare a Burzynski effective stress at yield to the tension yield test:

$$\sigma_B = \left(3(\kappa - 1)p + \sqrt{9(\kappa - 1)^2 p^2 + 4\kappa\sigma_M^2}\right)/2\kappa = k_t$$

If the material yield stress is the same in tension and compression ($\kappa = 1$) then the Burzynski criterion reduces to the von Mises criterion at impending failure:

$$\sigma_B(@\kappa = 1) = \left(0 + \sqrt{4\sigma_M^2}\right)/2 = \sigma_M = k_t$$

When the material yield stress is the same in tension and compression ($\kappa = 1$) then the Burzynski criterion reduces to the von Mises criterion. The Burzynski criterion can be extended to include the shear stress failure value, k_s. Then the criterion form is

$$\sigma_B^2 = k_c k_t \sigma_M\, 2/(3k_s^2) + (9 - 3\, k_c k_t/k_s^2)p^2 + 3(k_c - k_t)p = k_c k_t = \kappa\, k_t^2.$$

To see how to plot the Burzynski effective stress in SolidWorks simulation read the help file Burzynski_SWS_Plot_Instructions.pdf. Minor edits will create Hoffman and Drucker-Prandl effective stress plots.

Hoffman criterion:

The Hoffman effective stress criterion [Hoffman 1967] predicts the onset of yielding when

$$\frac{\sigma_M{}^2}{k_c k_t} + p\left(\frac{1}{k_t} - \frac{1}{k_c}\right) = 1$$

or

$$\sigma_M{}^2/\kappa k_t + p(1 - 1/\kappa) = k_t$$

so when $\kappa = 1$, $\sigma_M{}^2 = k_t^2$ as required from the tensile test.

Drucker-Prandtl criterion:

The Drucker-Prager effective stress criterion [Drucker 1952] predicts the onset of yielding of soils. When converted from soil mechanics notation to the current notation its form is

$$\sigma_M(k_c + k_t) + p(k_c - k_t) - 2k_c k_t = 0$$

or

$$[\sigma_M(\kappa + 1) + p(\kappa - 1)]/2\kappa = k_t$$

so when $\kappa = 1$, $2\sigma_M = 2k_t$ as required from the tensile test.

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