Rigid Body Dynamics Review

Kinematics of Relative Motion

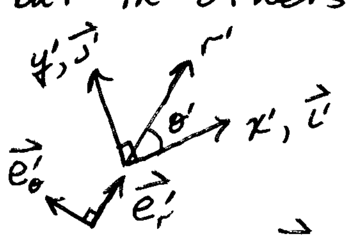
$$\left.\begin{aligned}
\vec{r}_A &= \vec{r}_B + \vec{r}_{A/B} \\
\vec{v}_A &= \vec{v}_B + \underbrace{\vec{\omega} \times \vec{r}_{A/B}}_{\vec{v}_{A/B}} \\
\vec{a}_A &= \vec{a}_B + \underbrace{\vec{\alpha} \times \vec{r}_{A/B} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{A/B})}_{\vec{a}_{A/B}}
\end{aligned}\right\}$$

For A and B attached to the same rigid body.

$$\vec{v}_P = \vec{v}_B + \vec{v}_{P/Body} + \vec{\omega} \times \vec{r}_{P/B}$$

$$\underbrace{\vec{a}_P = \vec{a}_B + \vec{a}_{P/Body} + 2\vec{\omega} \times \vec{v}_{P/Body} + \vec{\alpha} \times \vec{r}_{P/B} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/B})}_{\text{For P in motion with respect to the rigid body.}}$$

To analyze $\vec{v}_{P/Body}$ and $\vec{a}_{P/Body}$ consider a coordinate system attached to the rigid body and rotating with it. In some cases a Cartesian system is most appropriate, but in others a polar system is better.



$$\vec{v}_{P/Body} = \dot{x}'\vec{i} + \dot{y}'\vec{j} = \dot{r}'\vec{e}'_r + r'\dot{\theta}'\vec{e}'_\theta$$

$$\vec{a}_{P/Body} = \ddot{x}'\vec{i} + \ddot{y}'\vec{j} = (\ddot{r}' - r'\dot{\theta}'^2)\vec{e}'_r + (r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}')\vec{e}'_\theta$$

Kinetics of Planar motion

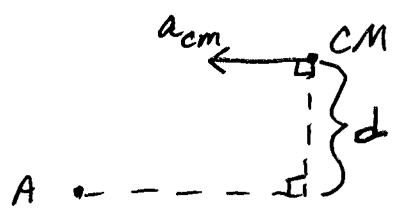
$$\vec{\omega} = \omega \vec{k} \, , \quad \vec{\alpha} = \alpha \vec{k}$$

$$\vec{F} = m \vec{a}_{cm} \quad \begin{cases} F_x = m\, a_{cm,x} \\ F_y = m\, a_{cm,y} \end{cases}$$

$$M_z^{cm} = I_{cm}\, \alpha$$

or $M_z^A = I_A \alpha$ if A is fixed ($\vec{v}_A = 0$) <u>and</u> attached to the body.

or $M_z^A = I_{cm}\alpha + m\, \underbrace{a_{cm} d}$ if A is fixed ($\vec{v}_A = 0$)



Anytime you see d with $m a_{cm}$ or $m v_{cm}$ it should be interpreted as a moment arm.

Parallel axis theorem: $I_A = I_{cm} + m\underbrace{d^2}$

Here d is simply the distance between A and the CM.

Energy Methods

$$KE_i + PE_i + W^{nc} = KE_f + PE_f$$

Work due to a couple: $W^c = \int_{\theta_i}^{\theta_f} C \, d\theta$
where C and θ are
positive in the counterclockwise direction.

Potetial energy of an angular spring:

$$U^{\theta\text{-}k} = PE^{\theta\text{-}k} = \frac{1}{2} k \, \Delta\theta^2$$
$$= \frac{1}{2} k \, (\theta - \theta_0)^2$$

where $\theta_0 \equiv$ "free angle" of the spring.

This can be derived from $W^c = \int_{\theta_i}^{\theta_f} C \, d\theta$
by noting that

$$C = k(\theta - \theta_0)$$

$$\Delta PE^{\theta\text{-}k} = \int_{\theta_i}^{\theta_f} C \, d\theta = \int_{\theta_i}^{\theta_f} k(\theta - \theta_0) \, d\theta$$
$$= \frac{1}{2} k (\theta - \theta_0)^2 \Big]_{\theta_i}^{\theta_f}$$
$$= \underbrace{\frac{1}{2} k (\theta_f - \theta_0)^2}_{PE_f} - \underbrace{\frac{1}{2} k (\theta_i - \theta_0)^2}_{PE_i}$$

Finally: $\boxed{KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2}$

or $\boxed{KE = \frac{1}{2} I_A \omega^2 \quad \text{if } A \text{ is the instant center}}$

I would only use this if there is an obvious fixed point in the problem.

Momentum Methods

Linear momentum: $\vec{p} = m \vec{v}_{cm}$

Angular momentum (planar motion)

$\rightarrow \vec{h} = h_z \vec{k}$

$$h_z^{cm} = I_{cm} \omega$$

d is the "moment arm" for $m\vec{v}_{cm}$

$$h_z^A = I_{cm} \omega + m v_{cm} d \rightarrow A \text{ is arbitrary}$$

$$h_z^A = I_A \omega \rightarrow A \text{ is the instant center of velocity}$$

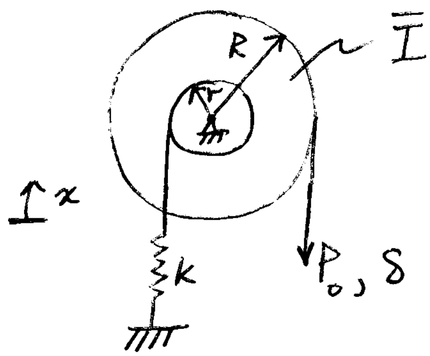
Conservation of linear momentum holds if the impulse of all external forces is zero.

$$\int_{t_i}^{t_f} \vec{F} \, dt = m \vec{v}_{cm}^{\,f} - m \vec{v}_{cm}^{\,i}$$

Angular impulse and momentum applies to the CM or a fixed point.

$$\boxed{\int_{t_i}^{t_f} M_z^A \, dt = h_z^{A,f} - h_z^{A,i} \qquad \begin{array}{l} A \text{ is fixed} \\ \text{or } A = CM \end{array}}$$

Example Problem 18.53



P_o = constant
Starts from rest.
Determine ω_{max}.

$$\cancelto{0}{KE_i} + \cancelto{0}{PE_i} + W^{nc} = KE_f + PE_f$$

$$P_o \delta = \tfrac{1}{2}\bar{I}\omega^2 + \tfrac{1}{2}kx^2$$

If the compound pulley rotates through an angle of $\Delta\theta$ then

$$\delta = R\Delta\theta \quad \text{and} \quad x = r\Delta\theta$$

$$\therefore \quad P_o R\Delta\theta = \tfrac{1}{2}\bar{I}\omega^2 + \tfrac{1}{2}kr^2\Delta\theta^2$$

$$\omega^2 = \frac{1}{\bar{I}}\left(2P_o R\Delta\theta - kr^2\Delta\theta^2\right)$$

Note that if we maximize ω^2 we also maximize ω.

$$\frac{d\omega^2}{d\Delta\theta} = \frac{1}{\bar{I}}\left(2P_0R - 2kr^2\Delta\theta\right) = 0$$

$$\therefore \quad \Delta\theta = \frac{P_0R}{kr^2}$$

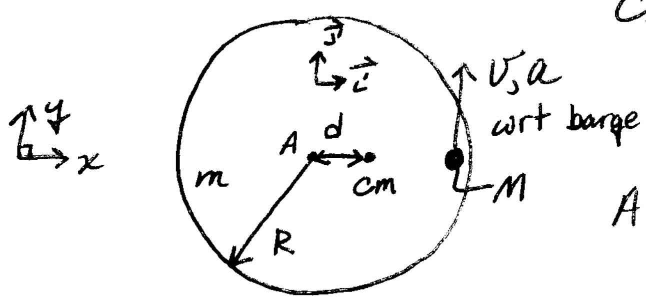
$$\rightarrow \omega_{max}^2 = \frac{1}{\bar{I}}\left(2P_0R\frac{P_0R}{kr^2} - kr^2\frac{P_0^2R^2}{k^2r^4}\right)$$

$$= \frac{1}{\bar{I}}\frac{P_0^2R^2}{kr^2}$$

$$\therefore \quad \boxed{\omega_{max} = \frac{P_0R}{r}\sqrt{\frac{1}{\bar{I}k}}}$$

Example: Large Marge on the Barge

Marge, mass M, decides to get some exercise by running around the perimeter of a circular barge of mass m and radius R floating on a lake. Initially the barge and Marge are at rest. At some instant in time Marge's tangential velocity wrt the barge is v and her tangential acceleration wrt the barge is a. At this instant determine α and ω of the barge. Neglect friction.



CM = center of mass of the system

A = center of mass of the barge

$$d = \frac{1}{M+m}(MR) = \frac{M}{M+m}R$$

Conservation of linear momentum implies that CM remains fixed wrt the lake.

Note A is not fixed or the CM of the system.

No external moments → conservation of angular momentum about CM of system.

Linear momentum $\rightarrow$ $0 = m\vec{v}_A + M\vec{v}_M$

In the position shown: $\vec{v}_M = \vec{v}_A + \vec{v}_{M/Body} + \vec{\omega} \times \vec{r}_{M/A}$

$$\vec{v}_M = \vec{v}_A + v\vec{j}' + \omega\vec{k} \times R\vec{i}$$

$\vec{j}' = \vec{j}$ in position shown

$$\vec{v}_m = \vec{v}_A + v\vec{j} + \omega R\vec{j}$$

$$v_{Mx}\vec{i} + v_{My}\vec{j} = v_{Ax}\vec{i} + v_{Ay}\vec{j} + (v + \omega R)\vec{j}$$

$$\therefore \quad v_{Mx} = v_{Ax}$$

$$v_{My} = v_{Ay} + v + \omega R$$

but CLM $\rightarrow$ $m v_{Ax} = -M v_{Mx} = -M v_{Ax}$

$$\therefore \quad v_{Ax} = 0 \rightarrow v_{Mx} = 0$$

and $\quad m v_{Ay} = -M v_{My} = -M(v_{Ay} + v + \omega R)$

$$\therefore \quad v_{Ay} = \frac{-M}{m+M}(v + \omega R)$$

$$\rightarrow v_{My} = \frac{m}{m+M}(v + \omega R)$$

$$v_{Ax} = v_{Mx} = 0$$

Angular momentum → $h_z^{cm,i} = h_z^{cm,f}$, $I = \frac{1}{2} m R^2$

$$\therefore \quad 0 = I\omega - m v_{Ay} d + M v_{My} (R-d)$$

$$= I\omega - m\left(\frac{-M}{m+M}\right)(v+\omega R)d$$

$$+ M\left(\frac{m}{m+M}\right)(v+\omega R)(R-d)$$

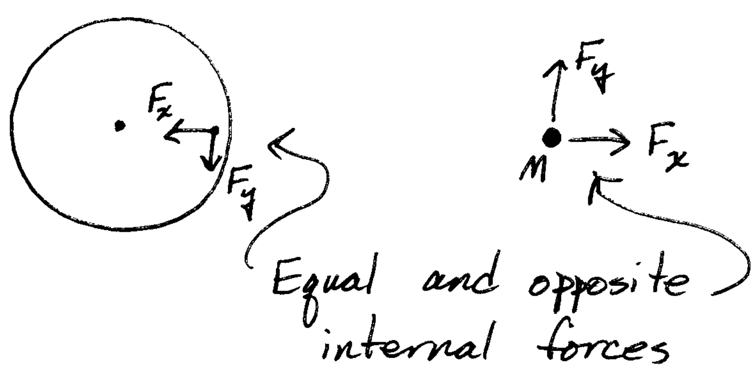
$$0 = I\omega + \frac{Mm}{m+M}(v+\omega R)R$$

$$0 = \left(I + \frac{Mm}{m+M}R^2\right)\omega + \frac{Mm}{m+M} v R$$

$$\therefore \quad \omega = -\frac{\frac{Mm}{m+M} R v}{I + \frac{Mm}{m+M} R^2}$$

$$\boxed{\omega = \frac{-MmRv}{\frac{1}{2}m(m+M)R^2 + MmR^2} = \frac{-Mm\,v}{\frac{1}{2}m^2 R + \frac{3}{2}mMR}}$$

<u>Accelerations</u> → use $\vec{F} = m\vec{a}$ methods



Marge: $F_x = M a_{Mx}$

$$F_y = M a_{My}$$

Barge: $-F_x = m a_{Ax}$

$$-F_y = m a_{Ay}$$

$$-F_y R = I \alpha$$

$$\vec{a}_M = \vec{a}_A + \vec{a}_{M/Barge} + 2\vec{\omega} \times \vec{v}_{M/Barge} + \vec{\alpha} \times \vec{r}_{M/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{M/A})$$

$$a_{Mx}\vec{i} + a_{My}\vec{j} = a_{Ax}\vec{i} + a_{Ay}\vec{j} + a\vec{j} - \frac{v^2}{R}\vec{i}$$
$$+ 2\omega\vec{k} \times v\vec{j} + \alpha\vec{k} \times R\vec{i}$$
$$+ \omega\vec{k} \times (\omega\vec{k} \times R\vec{i})$$

$$a_{Mx}\vec{i} + a_{My}\vec{j} = \left(a_{Ax} - \frac{v^2}{R} - 2\omega v - \omega^2 R\right)\vec{i}$$
$$+ \left(a_{Ay} + a + \alpha R\right)\vec{j}$$

We need α so we also need F_y.

$$F_y = M a_{My} = -m a_{Ay}$$

$$M(a_{Ay} + a + \alpha R) = -m a_{Ay}$$

$$\therefore \quad a_{Ay} = \frac{-M}{m+M}(a+\alpha R)$$

$$\rightarrow \quad F_y = \frac{mM}{m+M}(a+\alpha R)$$

Finally $\quad -F_y R = I\alpha$

$$-\frac{mM}{m+M}(a+\alpha R)R = I\alpha$$

$$\therefore \quad \boxed{\alpha = \frac{-\frac{mM}{m+M}aR}{I + \frac{mM}{m+M}R^2}}$$

We could have also found the x-force on Marge as well.

Summary:

$$\omega = \frac{-Mmv}{\frac{1}{2}m^2R + \frac{3}{2}mMR} = \frac{-2M}{m+3M}\frac{v}{R}$$

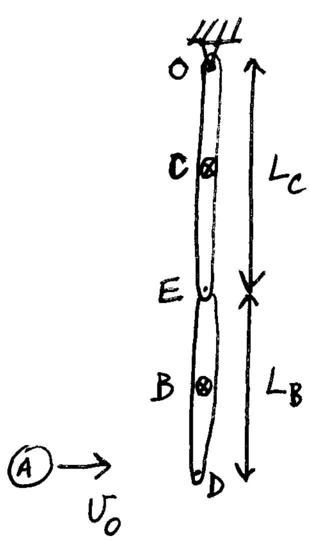
$$\alpha = \frac{-\frac{mM}{m+M}aR}{\frac{1}{2}mR^2 + \frac{mM}{m+M}R^2} = \frac{-2M}{m+3M}\frac{a}{R}$$

Rigid Body Impacts

- For the analysis of rigid body impacts it is useful to draw FBDs of all interacting bodies but with forces replaced by impulses of forces.

- Then, if we make the approximation that the duration of the impact is effectively zero, then the principle of angular impulse and angular momentum can be applied to any arbitrary point by treating them as fixed points not attached to a rigid body.

Example Problem 19.74

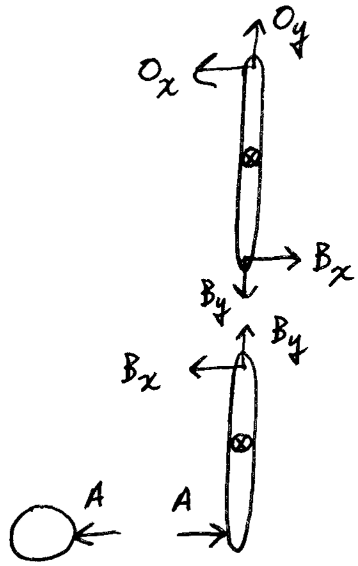


m_A, m_B, m_C given
L_C, L_B, v_0 given
coefficient of restitution e given
Determine ω_B and ω_C just after the impact.

$$e = v_{sep} / v_{App} = \frac{v_{Dx} - v_A}{v_0}$$

$$v_{Dx} = v_{Ex} + \omega_B L_B = \cancel{v_{0x}}^{\,0} + \omega_C L_C + \omega_B L_B$$

Impulsive Force FBDs



A, B_x, B_y, O_x, O_y are all <u>impulses</u> of forces.

There are many ways to solve this problem. I will go through many equations that are valid, but not all are needed.

– Linear impulse – momentum for each object.

A: $\quad -A = m_A v_A - m_A v_0$

B: $\quad A - B_x = m_B v_{Bx}$

$\quad B_y = m_B v_{By} = 0 \qquad \vec{v}_B = v_{Bx}\vec{i}$

(all points on the bars can only have $\vec{i}$ velocities at the instant shown)

C: $\quad B_x - O_x = m_c v_{cx}$

$\quad O_y - B_y = m_c v_{cy} = 0$

$\rightarrow B_y = O_y = 0$

– Angular impulse and angular momentum for each bar

bar B about its CM: $\quad A\frac{L_B}{2} + B_x\frac{L_B}{2} = \frac{1}{12} m_B L_B^2\, \omega_B$

bar B about the connection at E: $\quad A L_B = \frac{1}{12} m_B L_B^2\, \omega_B + m_B v_{Bx}\frac{L_B}{2}$

where $$v_{Bx} = v_{Ex} + \omega_B \frac{L_B}{2}$$

$$= \cancelto{0 \text{ (fixed for all time)}}{v_{Ox}} + \omega_C L_C + \omega_B \frac{L_B}{2}$$

$$\rightarrow v_{Bx} = \omega_C L_C + \omega_B \frac{L_B}{2}$$

also note that $v_{Cx} = \omega_C \frac{L_C}{2}$

bar C about its CM: $B_x \frac{L_C}{2} + O_x \frac{L_C}{2} = \frac{1}{12} m_C L_C^2 \omega_C$

bar C about point O: $B_x L_C = \underbrace{\frac{1}{3} m_C L_C^2 \omega_C}_{\text{valid b/c O is fixed in space and to the bar for all time.}}$

Unknowns: $A, B_x, B_y, O_x, O_y, v_A, \omega_B, \omega_C$
(v_{Bx} & v_{Cx} are given in terms of ω_B & ω_C already)

Equations:
3 x momentum eqs. for A, B, C
2 y momentum eqs for B, C $\rightarrow B_y = O_y = 0$
1 angular momentum eq. for B
1 angular momentum eq. for C
1 coefficient of restitution eq.

8 equations for the 8 unknowns

Using the other angular momentum equations for B & C would give redundant equations, (think x & y momentum + angular momentum).

This procedure has introduced many new unknowns. We can get around this by looking at systems of objects.

① Conservation of angular momentum of A, B & C about O

fixed in space and to bar C

$$m_A v_0 (L_B + L_C) = m_A v_A (L_B + L_C) + \frac{1}{3} m_C L_C^2 \omega_C + \frac{1}{12} m_B L_B^2 \omega_B + m_B v_{Bx} \left(L_C + \frac{L_B}{2}\right)$$

② Conservation of angular momentum of A & B about E

fixed in space but <u>not</u> to the bars.

$$m_A v_0 L_B = m_A v_A L_B + \frac{1}{12} m_B L_B^2 \omega_B + m_B v_{Bx} \frac{L_B}{2}$$

Use these two equations plus the coefficient of restitution equation to solve for v_A, ω_B, ω_C. (Note: v_{Dx}, v_{Bx} are given in terms of ω_B & ω_C)