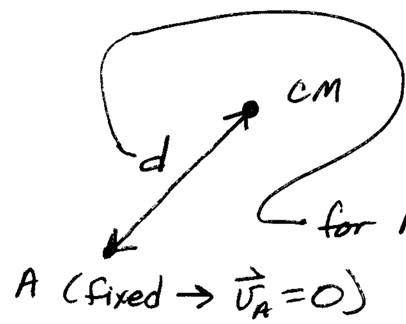
Review of Planar Rigid Body Kinetics

$$\vec{F} = m\vec{a}_{cm}$$

$$\vec{M}_A = \vec{r}_{cm/A} \times m\vec{a}_{cm} + I_{cm}\alpha\vec{k}$$ A is fixed or $A = CM$

Planar motion



for A fixed and attached to body.

$$F_x = m\,a_{cmx}$$

$$F_y = m\,a_{cmy}$$

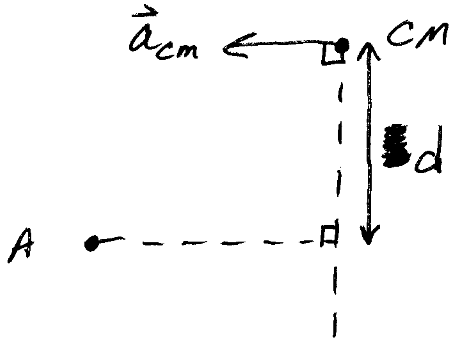
$$M_z^A = (I_{cm} + md^2)\alpha = I_A\alpha$$ A fixed and attached to the body. 2nd choice. Sometimes easiest to apply.

or $\underbrace{M_z^{cm} = I_{cm}\alpha}_{\text{Use this if you can.}}$ $A = CM$

If A is fixed, but not attached to the body then

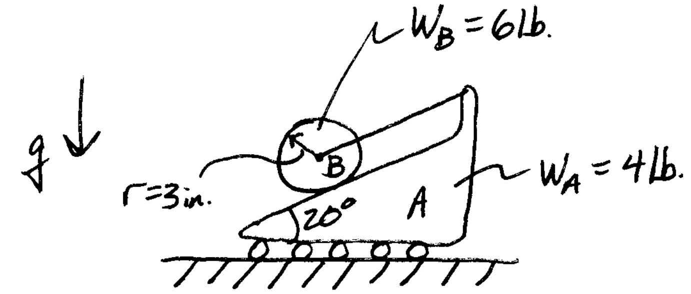
$$M_z^A = I_{cm}\alpha + m\,a_{cm}d$$ A fixed in space

Last resort use this.



(Think of d as the moment arm for $m\vec{a}_{cm}$)

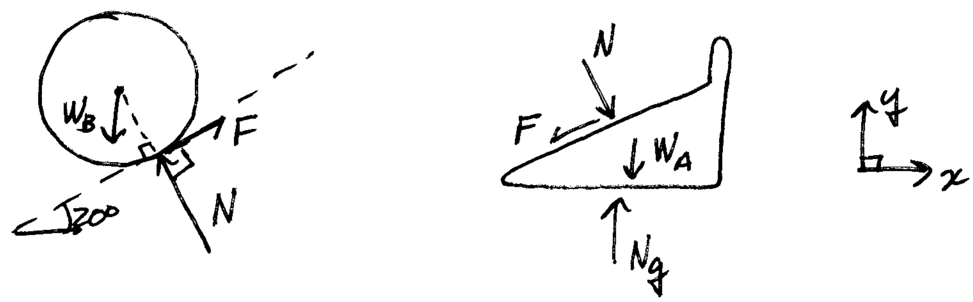
Example Problem



Initially at rest.
Rolls without slipping.
Neglect friction between wedge & ground.

Determine the acceleration of the wedge and the angular acceleration of the cylinder immediately after the cord is cut.

Draw free body diagrams.



Cylinder

$$① \ \Sigma F_x = F\cos 20° - N\sin 20° = m_B a_{Bx} = \frac{W_B}{g} a_{Bx}$$

$$② \ \Sigma F_y = F\sin 20° + N\cos 20° - W_B = m_B a_{By} = \frac{W_B}{g} a_{By}$$

$$③ \ \Sigma M_z^{cm} = F r = I_{cm}\alpha_B = \frac{1}{2} m_B r^2 \alpha_B = \frac{1}{2}\frac{W_B}{g} r^2 \alpha_B$$

Wedge

(4) $\sum F_x = -F\cos 20° + N\sin 20° = m_A a_{Ax} = \frac{W_A}{g} a_{Ax}$

(5) $\sum F_y = -F\sin 20° - N\cos 20° - W_A + N_g = m_A \cancel{a_{Ay}}^{0}$

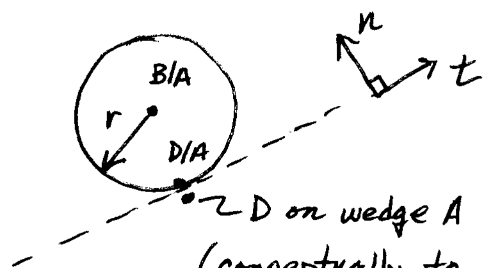
5 equations for $\underbrace{F, N, a_{Bx}, a_{By}, \alpha_B, a_{Ax}, N_g}_{7 \text{ unknowns}}$

Note that we could have analyzed the cylinder on an inclined coordinate system. Also note that analyzing the system as a whole will not yield any new equations.

We must now use relative acceleration and velocity equations to find relationships between a_{Bx}, a_{By}, α_B and a_{Ax}.

First, $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$

To analyze $\vec{a}_{B/A}$ we can think of the wedge as being fixed and the cylinder rolling down it. Let's be general and analyze both velocity and acceleration.



$t \equiv$ tangential direction

$n \equiv$ normal direction

(conceptually to analyze $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$ both $\vec{v}_D = 0$ and $\vec{a}_D = 0$)

$$\vec{v}_{D/A} = \vec{v}_{B/A} + \vec{v}_{D/B} = \vec{v}_{B/A} + \vec{\omega}_B \times \vec{r}_{D/B}$$

Since D/A is coincident with D <u>and</u> there is no slipping, both the normal and tangetial components of $\vec{v}_{D/A}$ equal those of $\vec{v}_D = 0$.

$$\therefore \quad \vec{v}_{B/A} = -\vec{\omega}_B \times \vec{r}_{D/B} \quad \text{for rolling w/o slipping}$$

or $\boxed{\vec{v}_{B/A} = -\omega_B \, r \, \vec{e}_t}$ rolling w/o slipping

acceleration: $\vec{a}_{D/A} = \vec{a}_{B/A} + \vec{\alpha}_B \times \vec{r}_{D/B} + \vec{\omega}_B \times (\vec{\omega}_B \times \vec{r}_{D/B})$

$$\vec{a}_{D/A} = \vec{a}_{B/A} + \alpha_B r \vec{e}_t + \omega_B^2 r \vec{e}_n$$

for rolling without slipping $\vec{a}_{D/A} = a_{D/A} \vec{e}_n$

(i.e. $a_{D/A}^{tangential} = 0$) and $\vec{a}_{B/A} = a_{B/A} \vec{e}_t$

$\therefore$ $\boxed{a_{B/A} = -\alpha_B r}$ for rolling w/o slipping

In tangential direction.

$$\therefore \quad \vec{a}_B = \vec{a}_A + (-\alpha_B r)\vec{e}_t$$

$$a_{Bx}\vec{i} + a_{By}\vec{j} = a_{Ax}\vec{i} - \alpha_B r(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})$$

$$\boxed{\begin{aligned} a_{Bx} &= a_{Ax} - \alpha_B r \cos 20° \\ a_{By} &= -\alpha_B r \sin 20° \end{aligned}}$$

(3) $\rightarrow \quad F = \frac{1}{2}\frac{W_B}{g} r \alpha_B$

(4) $\rightarrow \quad N \sin 20° = \frac{W_A}{g} a_{Ax} + F\cos 20° = \frac{W_A}{g} a_{Ax} + \frac{1}{2}\frac{W_B}{g} r\alpha_B \cos 20°$

(1) $\rightarrow \frac{1}{2}\frac{W_B}{\cancel{g}} r\alpha_B \cos 20° - \frac{W_A}{g} a_{Ax} - \frac{1}{2}\frac{W_B}{\cancel{g}} r\alpha_B \cos 20° = \frac{W_B}{g} a_{Bx}$

$$-\frac{W_A}{\cancel{g}} a_{Ax} = \frac{W_B}{\cancel{g}}(a_{Ax} - \alpha_B r \cos 20°)$$

$$\therefore \quad a_{Ax} = \frac{W_B}{W_A + W_B}\alpha_B r \cos 20°$$

(2) $\rightarrow \frac{1}{2}\frac{W_B}{g}\alpha_B r \sin 20° + \left(\frac{W_A}{g} a_{Ax} + \frac{1}{2}\frac{W_B}{g}\alpha_B r \cos 20°\right)\frac{\cos 20°}{\sin 20°} - W_B = \frac{W_B}{g} a_{By}$

Multiply by $\sin 20°$ and substitute in for a_{Ax} and a_{By}

$$\frac{1}{2}\frac{W_B}{g}\alpha_B r + \frac{W_A W_B}{W_A + W_B}\alpha_B r \cos^2 20° - W_B \sin 20° = \frac{-W_B}{g}\alpha_B r \sin^2 20°$$

$$\therefore \quad \alpha_B = \left[\frac{\frac{1}{2}\frac{W_B}{g} r + \frac{W_A W_B}{g(W_A+W_B)} r\cos^2 20° + \frac{W_B}{g} r \sin^2 20°}{W_B \sin 20°}\right]^{-1}$$

(Sorry, I solved for $1/\alpha_B$ accidently.)

Plugging in $W_A = 4\ \text{lb.}$, $W_B = 6\ \text{lb}$, $r = 3\ \text{in}$, $g = 32.2\ \frac{ft}{s^2}$

$\hookrightarrow r = \frac{1}{4}\ ft.$

We get $\alpha_B = 45.4\ \frac{rad}{s^2}$

$$\boxed{\vec{\alpha}_B = 45.4\ \vec{k}\ \frac{rad}{s^2}}$$

$$\therefore\ a_{Ax} = \frac{W_B}{W_A + W_B}(\alpha_B r)\cos 20° = 6.4\ ft/s^2$$

$$\boxed{\vec{a}_A = 6.4\ \vec{i}\ \frac{ft}{s^2}}$$

Work-Energy

Work of a couple in planar motion.

$$W^c = \int_{\theta_1}^{\theta_2} C\, d\theta$$

where C is a couple in the z-direction and θ increases in the counterclockwise direction about the z-axis.

Kinetic energy: Particle system $-$ $KE = \sum_{i=1}^{N} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$

$\rightarrow$ Rigid body $-$ $KE = \int_V \frac{1}{2} \rho\, \vec{v} \cdot \vec{v}\, dV$

Take A to be on the rigid body, then

$$\vec{v} = \vec{v}_A + \vec{\omega} \times \vec{r}_{/A}$$

$$KE = \frac{1}{2} \int_V \rho \left[\vec{v}_A \cdot \vec{v}_A + 2 \vec{v}_A \cdot \vec{\omega} \times \vec{r}_{/A} + \vec{\omega} \times \vec{r}_{/A} \cdot \vec{\omega} \times \vec{r}_{/A} \right] dV$$

$$= \frac{1}{2} \vec{v}_A \cdot \vec{v}_A \int_V \rho\, dV$$

$$+ \frac{1}{2} \cdot 2\, \vec{v}_A \cdot \vec{\omega} \times \int_V \rho\, \vec{r}_{/A}\, dV$$

$$+ \frac{1}{2} \int_V \rho\, \underbrace{(\vec{\omega} \times \vec{r}_{/A})}_{\vec{A}} \cdot \underbrace{\vec{\omega}}_{\vec{B}} \times \underbrace{\vec{r}_{/A}}_{\vec{C}}\, dV$$

Recall $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A}$

$$\therefore \quad \frac{1}{2}\int_V \rho\,(\vec{\omega}\times\vec{r}_{/A})\cdot(\vec{\omega}\times\vec{r}_{/A})\,dV$$

$$= \frac{1}{2}\int_V \rho\,\vec{\omega}\cdot\vec{r}_{/A}\times(\vec{\omega}\times\vec{r}_{/A})\,dV$$

$$= \frac{1}{2}\vec{\omega}\cdot\underbrace{\int_V \rho\,\vec{r}_{/A}\times(\vec{\omega}\times\vec{r}_{/A})\,dV}_{\vec{h}_A}$$

$$\therefore KE = \frac{1}{2}m\underbrace{\vec{v}_A\cdot\vec{v}_A}_{v_A^2} + \vec{v}_A\cdot\vec{\omega}\times\int_V \rho\,\vec{r}_{/A}\,dV$$

$$+ \frac{1}{2}\vec{\omega}\cdot\vec{h}_A$$

For planar motion either choose $A = CM$ or choose A to be the instant center, i.e $\vec{v}_A = 0$.

$$A = CM \rightarrow \quad \int_V \rho\,\vec{r}_{/A}\,dV = \int_V \rho\,\vec{r}_{/cm}\,dV = \int_V \rho\,\vec{r}\,'\,dV = 0$$

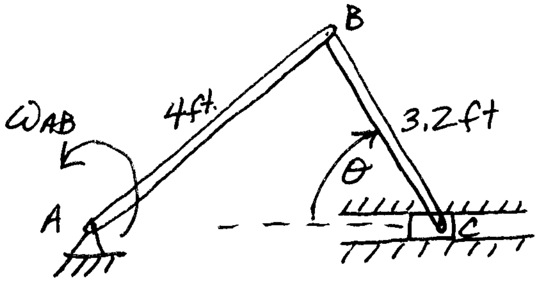
$$\vec{h}_A = \vec{h}_{cm} = I_{cm}\,\omega\,\vec{k}$$

$$\therefore \boxed{KE = \frac{1}{2}m v_{cm}^2 + \frac{1}{2}I_{cm}\,\omega^2}$$

or A fixed $\rightarrow \quad \vec{v}_A = 0 \rightarrow \vec{h}_A = I_A\,\omega\,\vec{k}$

$$\therefore \boxed{KE = \frac{1}{2}I_A\,\omega^2} \quad \underline{A \text{ fixed}}$$

Example Problem 18.21



$$m_{AB} = \frac{30\text{ lb.}}{32.2\text{ ft/s}^2}$$

$$m_{BC} = \frac{10\text{ lb}}{32.2\text{ ft/s}^2}$$

$$m_C = \frac{2\text{ lb.}}{32.2\text{ ft/s}^2}$$

$$\omega_{AB} = 2.4\ \frac{\text{rad}}{\text{s}}$$

Determine the kinetic energy of the entire system when $\theta = 50°$.

$$KE = \frac{1}{2} m_c v_c^2 + \underbrace{\frac{1}{2} I_A^{AB} \omega_{AB}^2}_{A \text{ is fixed}} + \frac{1}{2} m_{BC} v_{cm}^{BC\,2} + \frac{1}{2} I_{cm}^{BC} \omega_{BC}^2$$

Recall for a rod $\quad I_{cm} = \frac{1}{12} m L^2$

$$\therefore \quad I_{cm}^{BC} = \frac{1}{12} m_{BC} L_{BC}^2 = \frac{1}{12} \frac{10}{32.2} (3.2)^2 = 0.265$$

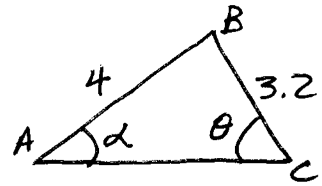
$$I_A^{AB} = I_{cm}^{AB} + m_{AB} d^2 = \frac{1}{3} m_{AB} L_{AB}^2 = \frac{1}{3} \frac{30}{32.2} 4^2 = 4.969$$

$$\therefore \quad \frac{1}{2} I_A^{AB} \omega_{AB}^2 = \frac{1}{2} 4.969 (2.4)^2 = 14.31 \text{ ft} \cdot \text{lb}$$

We still need v_c, v_{cm}^{BC}, ω_{BC}.

Use relative velocity equations.

$$\vec{v}_B = \cancelto{0}{\vec{v}_A} + \vec{\omega}_{AB} \times \vec{r}_{B/A}$$



$$\vec{v}_B = \omega_{AB}\vec{k} \times r_{B/A}(\cos\alpha\,\vec{i} + \sin\alpha\,\vec{j})$$

$$\frac{4}{\sin\theta} = \frac{3.2}{\sin\alpha}$$

$$\vec{v}_B = \omega_{AB}\, r_{B/A}(-\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j})$$

$$\rightarrow \alpha = \arcsin\left(\frac{3.2}{4}\sin 50°\right)$$

$$\alpha = 37.795°$$

$$\vec{v}_B = -2.4(4)\sin 37.8°\,\vec{i} + 2.4(4)\cos 37.8°\,\vec{j}$$
$$= (-5.883\,\vec{i} + 7.586\,\vec{j}) \text{ ft/s}$$

Then $\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$

$$\vec{v}_C = v_C\,\vec{i} = \omega_{AB}\, r_{B/A}(-\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j})$$
$$+ \omega_{BC}\vec{k} \times r_{C/B}(\cos\theta\,\vec{i} - \sin\theta\,\vec{j})$$
$$= \omega_{AB}\, r_{B/A}(-\sin\alpha\,\vec{i} + \cos\alpha\,\vec{j})$$
$$+ \omega_{BC}\, r_{C/B}(\sin\theta\,\vec{i} + \cos\theta\,\vec{j})$$

$$\therefore \vec{j} \quad 0 = \omega_{AB}\, r_{B/A}\cos\alpha + \omega_{BC}\, r_{C/B}\cos\theta$$

$$\omega_{BC} = \frac{-\omega_{AB}\, r_{B/A}\cos\alpha}{r_{C/B}\cos\theta} = -3.688 \frac{\text{rad}}{\text{s}}$$

$$\vec{i} \rightarrow v_C = -\omega_{AB}\, r_{B/A}\sin\alpha + \omega_{BC}\, r_{C/B}\sin\theta$$

$$v_C = -14.924 \text{ ft/s}$$

Finally we need v_{cm}^{BC}. We can get this by three different approaches.

$$\vec{v}_{cm}^{\,BC} = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{cm/B} \leftarrow 1$$

$$\vec{v}_{cm}^{\,BC} = \vec{v}_c + \vec{\omega}_{BC} \times \vec{r}_{cm/c} \leftarrow 2$$

$$2\vec{v}_{cm}^{\,BC} = \vec{v}_B + \vec{v}_c + \vec{\omega}_{BC} \times \underbrace{\left(\vec{r}_{cm/B} + \vec{r}_{cm/c}\right)}$$

Note $\vec{r}_{cm/B} = -\vec{r}_{cm/c}$

$\rightarrow \vec{r}_{cm/B} + \vec{r}_{cm/c} = 0$

$$\therefore\ \vec{v}_{cm}^{\,BC} = \frac{\vec{v}_B + \vec{v}_c}{2} \leftarrow 3$$

Any of these [redacted] 3 equations will give $\vec{v}_{cm}^{\,BC}$.

$$\vec{v}_{cm}^{\,BC} = \frac{1}{2}\left[(-5.883 - 14.924)\vec{i} + 7.586\vec{j}\right]$$

$$= -10.4035\vec{i} + 3.793\vec{j}$$

$$\left(v_{cm}^{BC}\right)^2 = 122.62\ \frac{ft^2}{s^2}$$

$$\therefore KE = \underbrace{14.31}_{\frac{1}{2}I_A^{AB}\omega_{AB}^2} + \underbrace{\frac{1}{2}\frac{2}{32.2}(14.924)^2}_{\frac{1}{2}m_c v_c^2} + \underbrace{\frac{1}{2}\ \overset{(0.265)}{[redacted]} \cdot (3.688)^2}_{\frac{1}{2}I_{cm}^{BC}\omega_{BC}^2} + \underbrace{\frac{1}{2}\frac{10}{32.2}\cdot 122.62}_{\frac{1}{2}m_{BC}\left(v_{cm}^{BC}\right)^2}$$

$$\boxed{KE = 42.07\ ft\cdot lb}$$