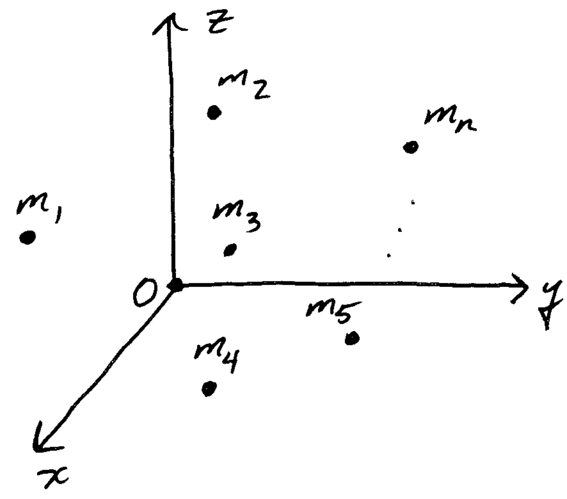
System Particle Dynamics



Take a system of m_n point masses. The problem that we would like to solve is, given some set of forces (external and internal) acting on this system of particles, and given the initial positions and velocities of the particles, what are the positions of these particles as a function of time.

For each particle we have $\Sigma \vec{F}_i = m_i \vec{a}_i$.

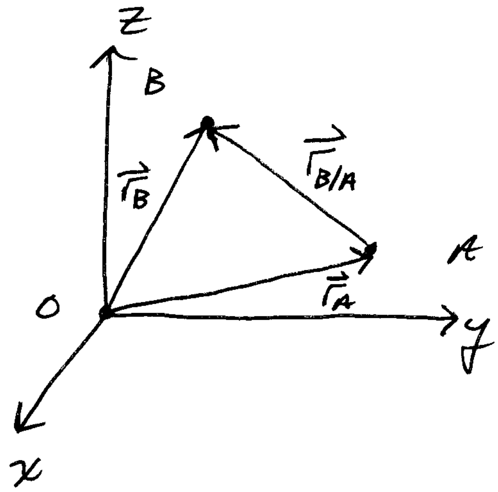
Where $\Sigma \vec{F}_i$ is the sum of both external and internal forces acting on mass m_i.

Internal forces arise due to interactions between two or more of the masses in the system.

We will get to the analysis of internal forces in more detail later, but now let's consider the kinematics of systems of particles.

Kinematics of relative motion.

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$



$$\vec{v}_B = \frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt}$$

$$= \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{a}_B = \frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{B/A}}{dt}$$

$$= \vec{a}_A + \vec{a}_{B/A}$$

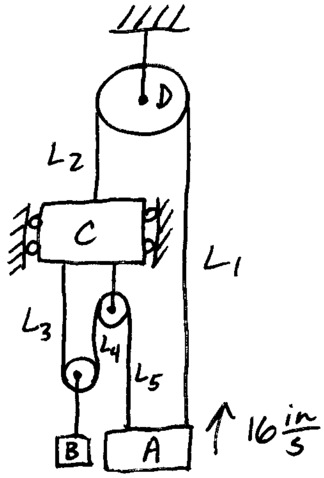
$\vec{r}_{B/A}$, $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$ are the relative position, velocity and acceleration of particle B with respect to A.

Kinematics of Constrained Motion

We have already encountered constrained motion in single particle dynamics with our rotating frame problem, recall $z = -r\cot\alpha$.

The role of constraint equations are to reduce the number of degrees of freedom in a system.

Example Problem 15.17



Determine v_B for the state shown.

Velocity relationships: (+ upward)

$$v_A = -\dot{L}_1 \quad (1)$$
$$v_C = -\dot{L}_2 \quad (2)$$
$$v_B = v_C - \dot{L}_3 \quad (3)$$
$$v_A = v_C - \dot{L}_5 \quad (4)$$

Length constraints:

$$L_1 + L_2 = \text{constant} \rightarrow \dot{L}_1 + \dot{L}_2 = 0 \quad (5)$$
$$L_3 + L_4 + L_5 = \text{constant} \rightarrow \dot{L}_3 + \dot{L}_4 + \dot{L}_5 = 0 \quad (6)$$
$$L_3 - L_4 = \text{constant} \rightarrow \dot{L}_3 = \dot{L}_4 \quad (7)$$
$$L_1 = L_2 + L_3 + L_5 - L_4 + \text{constant} \rightarrow \dot{L}_1 = \dot{L}_2 + \dot{L}_3 + \dot{L}_5 - \dot{L}_4 \quad (8)$$

(1), (2), (5) $\rightarrow v_C = -v_A$

$\rightarrow$ (4) $\rightarrow 2v_A = -\dot{L}_5$

(6), (7) $\rightarrow 2\dot{L}_4 + \dot{L}_5 = 0$

or $2\dot{L}_3 + \dot{L}_5 = 0$

$\rightarrow \dot{L}_3 = -\frac{1}{2}\dot{L}_5$

then (3) $\rightarrow$

$$v_B = -v_A + \tfrac{1}{2}\dot{L}_5$$
$$= -v_A + \tfrac{-1}{2}2v_A$$
$$= -2v_A$$

$$\therefore \boxed{v_B = -2v_A = -32\frac{in}{s}}$$

Back to our multiple particle system.

The position of the center of mass of a system of particles is the mass weighted average of the positions of each particle, i.e.

$$\vec{r}_{cm} = \bar{\vec{r}} = \frac{1}{m}\sum_{i=1}^{n} m_i \vec{r}_i \quad , \quad m = \sum_{i=1}^{n} m_i$$

then $$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{m}\sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt} = \frac{1}{m}\sum_{i=1}^{n} m_i \vec{v}_i$$

$$\vec{a}_{cm} = \frac{1}{m}\sum_{i=1}^{n} m_i \vec{a}_i$$

<u>Kinetics of a system of particles</u>

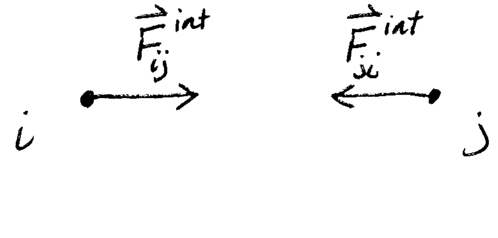
For each particle : [redacted] $\vec{F}_i = m_i \vec{a}_i$

[redacted] $$\vec{F}_i^{ext} + \sum_{j=1}^{n} \vec{F}_{ij}^{int} = m_i \vec{a}_i$$

Add together the equations for each particle

$$\sum_{i=1}^{n} \vec{F}_i^{ext} + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ij}^{int} = \sum_{i=1}^{n} m_i \vec{a}_i$$

Consider the second [redacted] term $\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ij}^{int}$



by Newton's Third Law
$$\vec{F}_{ji}^{int} = -\vec{F}_{ij}^{int}$$

Also note that this implies $\vec{F}_{ii}^{int} = -\vec{F}_{ii}^{int} = 0$,
i.e. the particle does not place a force on itself.

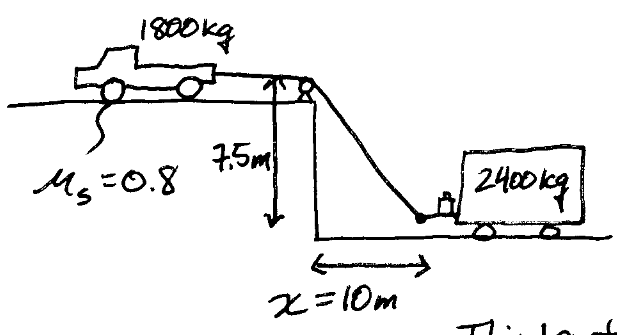
So, since $\vec{F}_{ij}^{int} = -\vec{F}_{ji}^{int} \rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ij}^{int} = 0$

i.e. all internal forces come in equal and opposite pairs and hence cancel each other out.

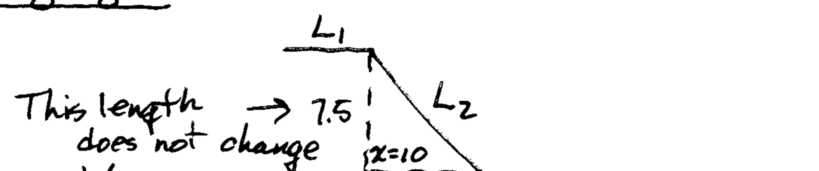
$$\therefore \quad \sum_{i=1}^{n} \vec{F}_i^{ext} = \sum_{i=1}^{n} m_i \vec{a}_i = m \underbrace{\frac{1}{m}\sum_{i=1}^{n} m_i \vec{a}_i}_{\vec{a}_{cm}}$$

$$\boxed{\sum_{i=1}^{n} \vec{F}_i^{ext} = m\,\vec{a}_{cm}}$$

<u>Example Problem 15.43</u>



What is max acceleration of the truck?



Constraint on cable length:

$$L_1 + L_2 = \text{constant} \rightarrow \dot{L}_1 + \dot{L}_2 = 0\,, \quad \ddot{L}_1 + \ddot{L}_2 = 0$$

$$L_2^2 = 7.5^2 + x^2 \rightarrow 2L_2\dot{L}_2 = 2x\dot{x} \rightarrow \dot{x} = \frac{L_2}{x}\dot{L}_2$$

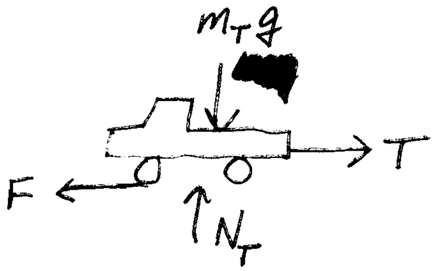
also from $L_2 \dot{L}_2 = x \dot{x}$ we can get

$$\dot{L}_2 \dot{L}_2 + L_2 \ddot{L}_2 = \dot{x}\dot{x} + x\ddot{x}$$

or $$\ddot{x} = \frac{1}{x}\left(L_2 \ddot{L}_2 + \dot{L}_2^2 - \dot{x}^2\right)$$

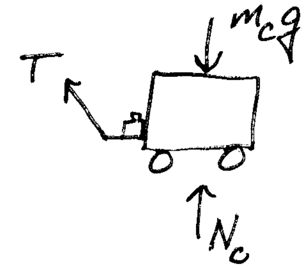
<u>FBD's</u>

Truck:



$\uparrow y \; \rightarrow x$

Camper:



For both truck and camper $a_y = 0$

Truck: $\Sigma F_y = -m_T g + N_T = 0 \rightarrow N_T = m_T g$

Camper: $\Sigma F_y = -m_c g + N_c + T \frac{7.5}{\sqrt{7.5^2+10^2}} = 0$

For truck $a_x^T = -\ddot{L}_1$
For camper $a_x^c = +\ddot{x}$

Truck: $\Sigma F_x = T - F = -m_T \ddot{L}_1 = m_T \ddot{L}_2$ from $\ddot{L}_1 + \ddot{L}_2 = 0$
Camper: $\Sigma F_x = -T \frac{10}{\sqrt{7.5^2+10^2}} = +m_c \ddot{x}$

Recall $\ddot{x} = \frac{1}{x}\left(L_2 \ddot{L}_2 + \dot{L}_2^2 - \dot{x}^2\right)$, $x = 10$, $L_2 = \sqrt{7.5^2+10^2}$
$\dot{L}_2 = 0$, $\dot{x} = 0$

$\therefore \ddot{x} = \frac{1}{10}\left(\sqrt{7.5^2+10^2}\, \ddot{L}_2\right) = 1.25 \ddot{L}_2$

$\therefore -T \frac{10}{\sqrt{7.5^2+10^2}} = +1.25 m_c \ddot{L}_2 \rightarrow T = -1.5625 m_c \ddot{L}_2$

$$\therefore \quad -1.5625\, m_c \ddot{L}_2 - F_{max} = +m_T \ddot{L}_2$$

$$F_{max} = \mu_s N_T = 0.8\, m_T g$$

$$(1.5625\, m_c + m_T)\ddot{L}_2 + 0.8\, m_T g = 0$$

$$\ddot{L}_2 = \frac{-0.8\, m_T g}{1.5625\, m_c + m_T} = -2.545\ \frac{m}{s^2}$$

$$\therefore \quad \boxed{a_x^T = -2.545\ \frac{m}{s^2}}$$

Maximum possible acceleration of the truck.

Recall $a_x^T = -\ddot{L}_1 = \ddot{L}_2$

The resulting acceleration of the camper would then be

$$a_x^c = +\ddot{x} = +1.25\ddot{L}_2 = -3.182\ \frac{m}{s^2}$$

and finally the tension in the cable would be

$$T = -1.5625\, m_c \ddot{L}_2 = 9545\ N$$

Particle System Energy

$$\vec{F}_i^{ext} + \sum_{j=1}^{n} \vec{F}_{ij}^{int} = m_i \frac{d\vec{v}_i}{dt}$$

As was shown for a single particle

$$\underbrace{\int \vec{F}_i^{ext} \cdot d\vec{r}_i + \sum_{j=1}^{n} \int \vec{F}_{ij}^{int} \cdot d\vec{r}_i}_{\text{Work done on the } i^{th} \text{ particle}} = \underbrace{\tfrac{1}{2} m_i v_{i2}^2 - \tfrac{1}{2} m_i v_{i1}^2}_{\text{Change in kinetic energy of the ith particle}}$$

Now sum over all particles in the system.

$$\underbrace{\sum_{i=1}^{n} \int \vec{F}_i^{ext} \cdot d\vec{r}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \int \vec{F}_{ij}^{int} \cdot d\vec{r}_i}_{\text{Work done by all forces in system}} = \underbrace{\sum_{i=1}^{n} \tfrac{1}{2} m_i \left(v_{i2}^2 - v_{i1}^2 \right)}_{\text{Change in total KE of system}}$$

Consider $\displaystyle \sum_{i=1}^{n} \sum_{j=1}^{n} \int \vec{F}_{ij}^{int} \cdot d\vec{r}_i$

$$= \int \vec{F}_{11}^{int} \cdot d\vec{r}_1 + \vec{F}_{12}^{int} \cdot d\vec{r}_1 + \ldots + \vec{F}_{1n}^{int} \cdot d\vec{r}_1$$

$$+ \int \vec{F}_{21}^{int} \cdot d\vec{r}_2 + \vec{F}_{22}^{int} \cdot d\vec{r}_2 + \ldots + \vec{F}_{2n}^{int} \cdot d\vec{r}_2$$

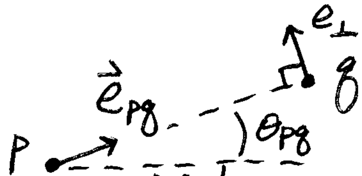
$$\vdots$$

$$+ \int \vec{F}_{n1}^{int} \cdot d\vec{r}_n + \vec{F}_{n2}^{int} \cdot d\vec{r}_n + \ldots + \vec{F}_{nn}^{int} \cdot d\vec{r}_n$$

Recall from Newton's 3rd Law: $\vec{F}_{ij}^{int} = -\vec{F}_{ji}^{int}$
and also $\vec{F}_{ii}^{int} = 0$

Hence, the internal work terms always come in pairs that look like

$$\int \vec{F}_{pq}^{int} \cdot d\vec{r}_p + \vec{F}_{qp}^{int} \cdot d\vec{r}_q = \int \vec{F}_{pq}^{int} \cdot d\vec{r}_p - \vec{F}_{pq}^{int} \cdot d\vec{r}_q = \int \vec{F}_{pq}^{int} \cdot \underbrace{(d\vec{r}_p - d\vec{r}_q)}_{\text{Relative Displacement increment/differential between } p \text{ and } q}$$



$$\vec{F}_{pq}^{int} = F_{pq} \vec{e}_{pq} = -\vec{F}_{qp}^{int}$$

$$d\vec{r}_{pq} = (d\vec{r}_p - d\vec{r}_q) = \vec{e}_{pq} \underbrace{d\ell_{pq}}_{\text{Change in length between } p \text{ and } q} + \vec{e}_{\perp} \underbrace{\ell_{pq} d\theta_{pq}}_{\text{Rotation of } pq \text{ line}}$$

$$\therefore \vec{F}_{pq}^{int} \cdot (d\vec{r}_p - d\vec{r}_q) = F_{pq} \vec{e}_{pq} \cdot (\vec{e}_{pq} d\ell_{pq} + \vec{e}_{\perp} \ell_{pq} d\theta_{pq})$$

$$= F_{pq} d\ell_{pq}$$

Therefore, if p and q are attached by some <u>rigid link</u> such that <u>$d\ell_{pq} = 0$</u> <u>and</u> we assume that $\vec{F}_{pq}^{int}$ acts along the line from p to q then the work done by the internal forces $\vec{F}_{pq}^{int}$ & $\vec{F}_{qp}^{int}$ is equal to zero.

In any case the work-energy principle remains unchanged.

$$\boxed{KE_i + PE_i + W^{nc} = KE_f + PE_f}$$

Where KE_i and KE_f account for the sum of the kinetic energies of all particles in the system, i.e.

$$KE = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

PE accounts for all sources of potential energy in the system including gravity & gravitation for <u>each</u> mass and the stored energy in all springs.

Finally, W^{nc} accounts for the work done by any and all forces that have not been accounted for in PE.

<u>Linear Momentum of a System</u>

$$\vec{F}_i^{\,ext} + \sum_{j=1}^{n} \vec{F}_{ij}^{\,int} = m_i \frac{d\vec{v}_i}{dt}$$

$$\underbrace{\int \vec{F}_i^{\,ext}\,dt + \sum_{j=1}^{n} \int \vec{F}_{ij}^{\,int}\,dt}_{\text{Impulse of the forces acting on particle } i} = \underbrace{m_i \vec{v}_{i2} - m_i \vec{v}_{i1}}_{\text{Change in momentum of particle } i}$$

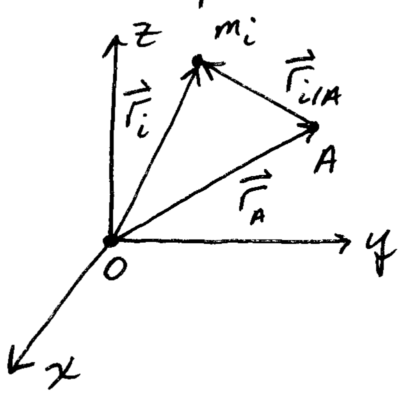
Sum over all particles

$$\sum_{i=1}^{n} \int \vec{F}_i^{\,ext} dt + \sum_{i=1}^{n}\sum_{j=1}^{n} \int \vec{F}_{ij}^{\,int} dt = \sum_{i=1}^{n} m_i (\vec{v}_{i2} - \vec{v}_{i1})$$

but since $\vec{F}_{ij}^{\,int} = -\vec{F}_{ji}^{\,int} \rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ij}^{\,int} = 0$

$$\therefore \boxed{\underbrace{\sum_{i=1}^{n} \int \vec{F}_i^{\,ext} dt}_{\text{Impulse of external forces on the system}} = \underbrace{\sum_{i=1}^{n} m_i (\vec{v}_{i2} - \vec{v}_{i1})}_{\text{Change in linear momentum of the system}}}$$

Angular Momentum of a System



$$\vec{r}_i = \vec{r}_A + \vec{r}_{i/A}$$

$$\vec{v}_i = \vec{v}_A + \vec{v}_{i/A}$$

$$\vec{F}_i^{\,ext} + \sum_{j=1}^{n} \vec{F}_{ij}^{\,int} = m_i \frac{d\vec{v}_i}{dt}$$

$$\vec{r}_{i/A} \times \vec{F}_i^{\,ext} + \sum_{j=1}^{n} \vec{r}_{i/A} \times \vec{F}_{ij}^{\,int} = \vec{r}_{i/A} \times m_i \frac{d\vec{v}_i}{dt}$$

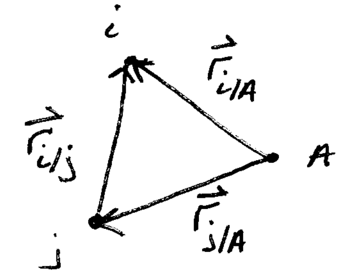
Sum over all particles

$$\sum_{i=1}^{n} \vec{r}_{i/A} \times \vec{F}_i^{ext} + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_{i/A} \times \vec{F}_{ij}^{int} = \sum_{i=1}^{n} \vec{r}_{i/A} \times m \frac{d\vec{v}_i}{dt}$$

Consider terms in $\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_{i/A} \times \vec{F}_{ij}^{int}$

from Newton's 3rd these always come in pairs like

$$\vec{r}_{i/A} \times \vec{F}_{ij}^{int} + \vec{r}_{j/A} \times \vec{F}_{ji}^{int}$$



$$= (\vec{r}_{i/A} - \vec{r}_{j/A}) \times \vec{F}_{ij}^{int}$$

$$= \vec{r}_{i/j} \times \vec{F}_{ij}^{int}$$

but $\vec{r}_{i/j} = r_{i/j}\,\vec{e}_{ij}$ and if the internal forces act along the line from i to j then

$$\vec{F}_{ij}^{int} = F_{ij}^{int}\,\vec{e}_{ij}$$

$$\therefore \quad \vec{r}_{i/j} \times \vec{F}_{ij}^{int} = r_{i/j} F_{ij}^{int} \overbrace{(\vec{e}_{ij} \times \vec{e}_{ij})}^{0} = 0$$

$$\therefore \quad \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_{i/A} \times \vec{F}_{ij}^{int} = 0$$

$$\rightarrow \quad \sum_{i=1}^{n} \vec{r}_{i/A} \times \vec{F}_i^{ext} = \sum_{i=1}^{n} \vec{r}_{i/A} \times m \frac{d\vec{v}_i}{dt}$$

Recall that $\frac{d}{dt}\left(\vec{r}_{i/A} \times \vec{v}_i\right) = \frac{d\vec{r}_{i/A}}{dt} \times \vec{v}_i$

$$+ \vec{r}_{i/A} \times \frac{d\vec{v}_i}{dt}$$

$$= \vec{v}_{i/A} \times \vec{v}_i + \vec{r}_{i/A} \times \frac{d\vec{v}_i}{dt}$$

$$= \underbrace{\vec{v}_i \times \vec{v}_i}_{0} - \vec{v}_A \times \vec{v}_i + \vec{r}_{i/A} \times \frac{d\vec{v}_i}{dt}$$

$$\therefore \vec{r}_{i/A} \times m_i \frac{d\vec{v}_i}{dt} = \frac{d}{dt}\left(\vec{r}_{i/A} \times m_i \vec{v}_i\right) + \vec{v}_A \times m_i \vec{v}_i$$

So $$\underbrace{\sum_{i=1}^{n} \vec{r}_{i/A} \times \vec{F}_i^{ext}}_{\vec{M}_A^{ext}} = \sum_{i=1}^{n} \frac{d}{dt}\underbrace{\left(\vec{r}_{i/A} \times m_i \vec{v}_i\right)}_{\vec{h}_{Ai}} + \sum_{i=1}^{n} \vec{v}_A \times m_i \vec{v}_i$$

$$\vec{M}_A^{ext} = \sum_{i=1}^{n} \frac{d\vec{h}_{Ai}}{dt} + \vec{v}_A \times m \underbrace{\frac{1}{m} \sum_{i=1}^{n} m_i \vec{v}_i}_{\vec{v}_{cm}}$$

$$\vec{M}_A^{ext} = \sum_{i=1}^{n} \frac{d\vec{h}_{Ai}}{dt} + \vec{v}_A \times m \vec{v}_{cm}$$

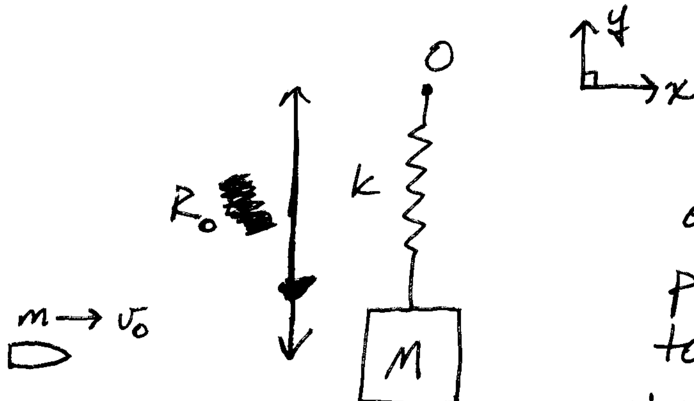
This equation implies there are two special types of points for analyzing angular momentum; a fixed point with $\vec{v}_A = 0$ or the CM with $\vec{v}_A = \vec{v}_{cm}$.

Then with either $\vec{v}_A = 0$ or $\vec{v}_A = \vec{v}_{cm}$ the term $\vec{v}_A \times m\vec{v}_{cm} = 0$ and we have

$$\vec{M}_A^{ext} = \sum_{i=1}^{n} \frac{d\vec{h}_{Ai}}{dt} \quad , \; A \text{ fixed or CM}$$

$$\boxed{\int_{t_1}^{t_2} \vec{M}_A^{ext}\, dt = \sum_{i=1}^{n} \left(\vec{h}_{Ai2} - \vec{h}_{Ai1}\right)}$$

Example Problem



The block of mass M is resting on a friction less plane and is attached to a spring with spring constant k, ie. $F^{spring} = k\delta$.

A bullet of mass m is fired into the block with an initial velocity of v_0 and embeds itself in the block. What is the maximum deflection of the spring? Is energy conserved? If not, where did it go and how much of it went?

Just before and just after the impact momentum is conserved in the x-direction because the spring can only apply a force in the y-direction in the initial position shown. Also note that after the impact both m and M have the same velocity.

$\therefore$ Momentum in $+x \rightarrow m v_0 = (m+M) v_i$

$$\boxed{v_i = \frac{m}{m+M} v_0}$$

The only external force on the 2 masses is that due to the spring. However, the spring force acts through point O (a fixed point), therefore angular momentum of the system is conserved about point O in the z-direction.

Angular momentum in $z \rightarrow m v_0 R_0 = (m+M) R v_\theta$

$$m v_0 R_0 = (m+M) R (R\dot{\theta})$$

$$m v_0 R_0 = (m+M) R^2 \dot{\theta}$$

After the collision energy is conserved.

$$KE_i + PE_i = KE_f + PE_f \quad , \quad W^{NC} = 0$$

$$\frac{1}{2}(m+M)\left(\frac{m}{m+M}v_0\right)^2 + 0 = \frac{1}{2}(m+M)(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{1}{2}k(R-R_0)^2$$

Maximum deflection of spring occurs when $\dot{R} = 0$.

$$\therefore \frac{1}{2}\frac{m^2}{m+M}v_0^2 = \frac{1}{2}(m+M)R^2\dot{\theta}^2 + \frac{1}{2}k(R-R_0)^2$$

2 unknown $R, \dot{\theta}$ with CAM $\rightarrow \dot{\theta} = \frac{m}{m+M}\frac{v_0 R_0}{R^2}$

$$\dot{\theta}^2 = \frac{m^2}{(m+M)^2}\frac{v_0^2 R_0^2}{R^4}$$

$$CE \rightarrow \frac{m^2}{m+M}v_0^2 = (m+M)\frac{m^2}{(m+M)^2}\frac{v_0^2 R_0^2}{R^2} + k(R-R_0)^2$$

$$\frac{m^2}{m+M}v_0^2 = \frac{m^2}{m+M}v_0^2\left(\frac{R_0}{R}\right)^2 + kR_0^2\left[\left(\frac{R}{R_0}\right)-1\right]^2$$

$$\left(\frac{R}{R_0}\right)^2 = 1 + \underbrace{\frac{kR_0^2(m+M)}{m^2 v_0^2}}_{A}\left(\frac{R}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)-1\right]^2$$

$$\left(\frac{R}{R_0}\right)^2 - 1 = A\left(\frac{R}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)-1\right]^2$$

$$\left[\left(\frac{R}{R_0}\right)-1\right]\left[\left(\frac{R}{R_0}\right)+1\right] = A\left[\left(\frac{R}{R_0}\right)-1\right]^2\left(\frac{R}{R_0}\right)^2$$

$$\left(\frac{R}{R_0}+1\right) = A\left(\frac{R}{R_0}-1\right)\left(\frac{R}{R_0}\right)^2 \quad , \text{ or } \frac{R}{R_0} = 1$$

A represents a constant with all known quantities in our problem. Let's take a specific case with $A = \frac{3}{4}$.

$$\left(R/R_0 + 1\right) = \frac{3}{4}\left(\frac{R}{R_0} - 1\right)\left(\frac{R}{R_0}\right)^2$$

$$\therefore \frac{3}{4}\left(\frac{R}{R_0}\right)^3 - \frac{3}{4}\left(\frac{R}{R_0}\right)^2 - \left(\frac{R}{R_0}\right) - 1 = 0$$

$$\underbrace{\left(\frac{R}{R_0} - 2\right)}_{\frac{R}{R_0} = 2} \underbrace{\left[\frac{3}{4}\left(\frac{R}{R_0}\right)^2 + \frac{3}{4}\frac{R}{R_0} + \frac{1}{2}\right]}_{\frac{R}{R_0} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{3}{2}}}{\frac{3}{2}}} = 0$$

$\left.\frac{R}{R_0} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{3}{2}}}{\frac{3}{2}}\right\}$ 2 Imaginary Roots

$\therefore$ Max deflection occurs at $\frac{R}{R_0} = 2$

Is Energy conserved from before the collision?

$$KE_{\text{Before}} : \frac{1}{2} m v_0^2$$

$$KE_{\text{after}} : \frac{1}{2}(m+M) v_i^2 = \frac{1}{2}\frac{m^2}{m+M} v_0^2$$

$$\frac{m^2}{m+M} = m \underbrace{\frac{m}{m+M}}_{<1} < m$$

$\therefore$ Energy is not conserved before and after the collision, it is lost in the impact due to internal frictional forces.