Work and Energy

The increment of work, dW, done by the force $\vec{F}$ is defined as

$$dW = \vec{F} \cdot d\vec{r}$$

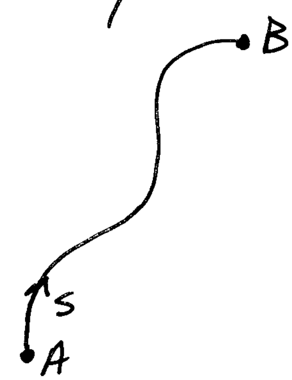
where $d\vec{r}$ is the displacement increment of the force.

Note that work is a scalar quantity.

Also note that a force can only do work if its line of action is partially in the same direction as its dislacement.

If a force moves along some path from A to B then the total work done by the force is given by

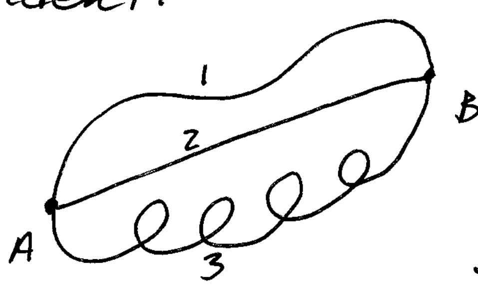
$$W_{AB} = \int_A^B \vec{F} \cdot d\vec{r}$$



Note that at any point along the path $d\vec{r} = \vec{e}_t \, ds$ where, as defined for path coordinates, $\vec{e}_t$ is a unit vector in the path direction.

- Conservative versus non-conservative forces.

Definition: A conservative force is any forces where the work done by the forces is dependent only on the position of the endpoints of the path but not on the shape of the path itself, i.e. the work done by a conservative force is path independent.



For a conservative force the work done would be identical for paths 1, 2 and 3.

$$W_{A\text{-}B} = \int_A^B \vec{F} \cdot d\vec{r}$$

$$= \int_A^B F_x\, dx + F_y\, dy + F_z\, dz$$

In order for this to be path independent we must be able to write the following,

$$F_x = \frac{-\partial V}{\partial x}, \quad F_y = \frac{-\partial V}{\partial y}, \quad F_z = \frac{-\partial V}{\partial z}$$

where the minus sign has been added for our physical interpretation.

Here V is a potential (soon we will call it the potential energy).

now $$W_{A-B}^{cons} = -\int_A^B \underbrace{\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz}_{\text{This is the total differential } dV}$$

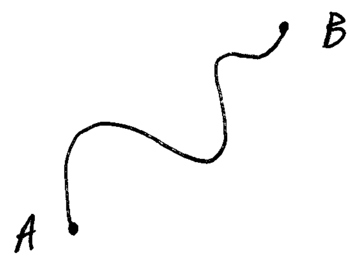
$$\boxed{W_{A-B}^{cons} = -\int_A^B dV = -V\Big]_A^B = V_A - V_B}$$

Some examples of $V \rightarrow V_g = mgh$,

$$V_{spring} = \frac{1}{2}k(L-L_0)^2, \quad V_G = \frac{-Gm_1m_2}{r}, \ldots$$

Is friction a conservative force?

$$W_{A-B}^{Friction} = \int_A^B \vec{F}\cdot d\vec{r}$$



Recall $d\vec{r} = \vec{e}_t\, ds$

$\vec{F}$ acts in the opposite direction of the motion, i.e. $\vec{F} = -F\vec{e}_t$ where $F>0$.

then $$W_{A-B}^{Friction} = \int_A^B -F\,\vec{e}_t\cdot\vec{e}_t\, ds$$

$$\rightarrow \boxed{W_{A-B}^{\text{Friction}} = -\int_A^B F\, ds}$$

Is this path independent? No.

Consider a constant friction force F. Then

$$W_{A-B}^{\text{Friction}} = -F \underbrace{\int_A^B ds}_{\text{Arclength from A to B does depend on the path!}} = \underbrace{-F \cdot \text{arclength}}_{\text{This result only applies to a constant friction force.}}$$

Note that since $F>0$ over the entire path that $W_{A-B}^{\text{Friction}} < 0$, i.e. the work done by friction is always negative.

- Back to conservative forces.

Gravity close to the earth: $\vec{F}_g = -mg\vec{k}$
(z-direction up from ground)

$$d\vec{r} = \vec{\imath}\,dx + \vec{\jmath}\,dy + \vec{k}\,dz$$

$$W_{A-B}^{\text{Gravity}} = \int_A^B -mg\vec{k} \cdot (\vec{\imath}\,dx + \vec{\jmath}\,dy + \vec{k}\,dz)$$

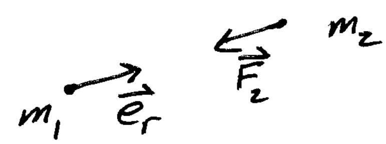
$$= \int_A^B -mg\,dz = -mgz\Big]_A^B$$

$$W_{A-B}^{Gravity} = \underbrace{-mgz_B}_{-V_B} + \underbrace{mgz_A}_{V_A}$$

In general we don't have to choose gravity to act in the z-direction so we usually write

$$\boxed{V^{Gravity} = mgh}$$ where h is the height above some arbitrarily chosen datum.

Gravitation: $\vec{F}_2 = -\dfrac{Gm_1m_2}{r^2}\vec{e}_r$



Taking m_1 to be the origin we can write

$$\boxed{d\vec{r} = \vec{e}_r\,dr + r\,\vec{e}_\theta\,d\theta}$$ Polar coordinate $d\vec{r}$

then
$$W_{A-B}^{Gravitation} = \int_A^B -\frac{Gm_1m_2}{r^2}\vec{e}_r \cdot (\vec{e}_r dr + r\vec{e}_\theta d\theta)$$

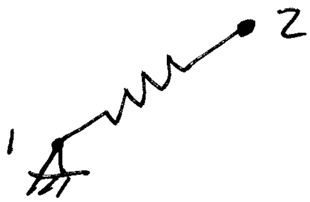
$$= \int_A^B -\frac{Gm_1m_2}{r^2}\,dr$$

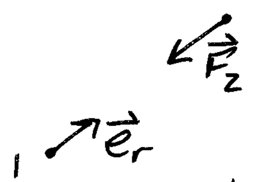
$$= \frac{Gm_1m_2}{r}\Big]_A^B$$

$$= \underbrace{\frac{Gm_1m_2}{r_B}}_{-V_B} - \underbrace{\frac{Gm_1m_2}{r_A}}_{V_A}$$

$$\therefore \boxed{V^{Gravitation} = \frac{-Gm_1m_2}{r}}$$

Springs:





Take point 1 to be our origin. Then the force that the spring places on point 2 is

$$\vec{F}_2 = -k\,\delta\,\vec{e}_r$$ where δ is the change in length of the spring, i.e.

$$\delta = r - r_0 \quad \text{with} \quad r_0 = \text{free length}$$

then again $d\vec{r} = \vec{e}_r\,dr + r\,\vec{e}_\theta\,d\theta$

$$\begin{aligned} W_{A-B}^{\text{spring}} &= \int_A^B -k(r-r_0)\,\vec{e}_r \cdot (\vec{e}_r\,dr + \vec{e}_\theta\, r\,d\theta) \\ &= \int_A^B -k(r-r_0)\,dr \\ &= -\tfrac{1}{2}k(r-r_0)^2 \Big]_A^B \\ &= \underbrace{-\tfrac{1}{2}k(r_B-r_0)^2}_{-V_B} + \underbrace{\tfrac{1}{2}k(r_A-r_0)^2}_{V_A} \end{aligned}$$

$$\therefore \boxed{V^{\text{spring}} = \tfrac{1}{2}k(r-r_0)^2 = \tfrac{1}{2}k\,\delta^2}$$

Work - Kinetic Energy

$$\frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = m\vec{a} \cdot \frac{d\vec{r}}{dt}$$

$$\vec{F} \cdot \vec{v} = m\frac{d\vec{v}}{dt} \cdot \vec{v}$$

consider $\frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2\vec{v} \cdot \frac{d\vec{v}}{dt}$

$$\therefore \quad \vec{F} \cdot \vec{v} = \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v})$$

$$(\vec{F}^{c} + \vec{F}^{NC}) \cdot \vec{v} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right)$$

$$\frac{dW^c}{dt} + \frac{dW^{NC}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right)$$

$$\int_{t_A}^{t_B} \frac{d}{dt}(W^c + W^{NC})\,dt = \int_{t_A}^{t_B} \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt$$

$$W^{c}_{A\text{-}B} + W^{NC}_{A\text{-}B} = \underbrace{\frac{1}{2} m v_B^2}_{T_B} - \underbrace{\frac{1}{2} m v_A^2}_{T_A}$$

$$V_A - V_B + W^{NC}_{A\text{-}B} = T_B - T_A$$

$$\boxed{T_A + V_A + W^{NC}_{A\text{-}B} = T_B + V_B}$$

or

$$\boxed{KE_i + PE_i + W^{NC} = KE_f + PE_f}$$

MECH 211 Supplemental Notes

Another approach to the work-kinetic energy derivation.

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

$$W_{A-B} = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B m\frac{d\vec{v}}{dt}\cdot d\vec{r} \leftarrow$$ This is the most natural starting point

Note : $d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}\,dt$

$$\rightarrow \frac{d\vec{v}}{dt}\cdot d\vec{r} = \vec{v}\cdot\frac{d\vec{v}}{dt}dt$$

Next consider $\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}$

$$= 2\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$\therefore \vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)$$

$$\rightarrow m\frac{d\vec{v}}{dt}\cdot d\vec{r} = m\vec{v}\cdot\frac{d\vec{v}}{dt}dt$$

$$= m\frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)dt$$

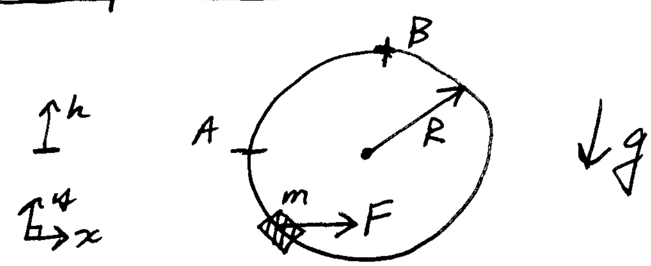
$$\therefore W_{A-B} = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right)dt$$

$$= \frac{1}{2}m\vec{v}\cdot\vec{v}\,\Big]_A^B .$$

$$\boxed{W_{A-B} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2}$$

In general it is wise to analyze only gravity, gravitation and spring forces within PE and all other forces within W^{NC}.

Example Problem 13.55



m starts from rest at A, determine the minimum constant F required to reach B

$$KE_i + PE_i + W^{NC} = KE_f + PE_f$$

$$0 + 0 + W^{NC} = 0 + m\underset{\sim}{g} R$$

$$W^{NC} = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B F\vec{i} \cdot (\vec{i}\,dx + \vec{j}\,dy)$$

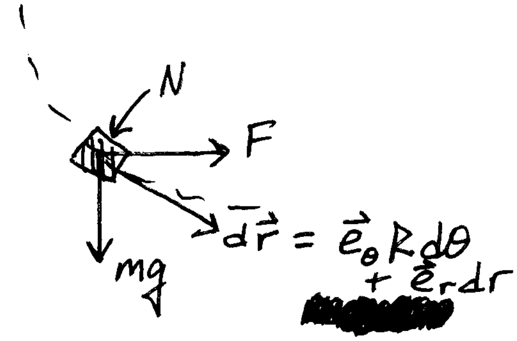
$$= \int_A^B F\,dx$$

$$= F(x_B - x_A) = FR$$

$$\therefore \quad FR = m\underset{\sim}{g} R$$

$$\boxed{F = m\underset{\sim}{g}}$$

Why did we neglect the normal force due to the rod when analyzing W^{NC}?



We analyze mg within the potential energy due to gravity. On the previous page we analyzed $\vec{F}$ with cartesian coordinates within [redacted] W^{nc}.

Why did we forget about N?

In the analysis of F we claimed $d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$. This was OK because F only acts in the x direction at all [redacted] locations along the path. However N changes along the path so we need to be more specific.

i.e. $\quad N = N\vec{e}_r \quad$ and $\quad d\vec{r} = \vec{e}_r\, dr + \vec{e}_\theta R\, d\theta$

but along our path $dr = 0$

$\therefore\ d\vec{r} = \vec{e}_\theta R\, d\theta$

$$\therefore\ W^N_{A-B} = \int_A^B N\vec{e}_r \cdot \left(\vec{e}_\theta R\, d\theta\right)$$

$$\vec{e}_r \cdot \vec{e}_\theta = 0$$

$$\therefore\ W^N_{A-B} = 0$$

Or if we did not realize $dr = 0$ we could get:

$$W^N_{A-B} = \int_A^B N\vec{e}_r \cdot \left(\vec{e}_r\, dr + \vec{e}_\theta R\, d\theta\right) = \int_A^B N\, dr = N(r_B - r_A) = N(R - R) = 0$$

Work-Energy Recap

$$W_{A-B} = \int_A^B \vec{F} \cdot d\vec{r}$$

$$\boxed{\begin{array}{lll} d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz & & \text{Cartesian} \\ d\vec{r} = \vec{e}_t\,ds & & \text{Path} \\ d\vec{r} = \vec{e}_r\,dr + \vec{e}_\theta\,r\,d\theta & & \text{Polar} \end{array}}$$

$$W_{A-B}^{cons} = V_A - V_B$$

$$V^{gravity} = mgh$$

$$V^{Gravitation} = -\frac{G m_1 m_2}{r}$$

$$V^{Spring} = \frac{1}{2} k \delta^2 = \frac{1}{2} k (r - r_0)^2 = \frac{1}{2} k (L - L_0)^2$$

$$V^{\theta\text{-spring}} = \frac{1}{2} k (\theta - \theta_0)^2$$

Work - Energy $\rightarrow$ $\boxed{KE_i + PE_i + W^{NC} = KE_f + PE_f}$

If there are no non-conservative forces acting on the particle then we say that the $\underbrace{\text{energy}}_{\text{i.e. kinetic + potential energy}}$ of the particle is conserved.

Impulse and Momentum

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$\int_{t_i}^{t_f} \vec{F}\,dt = \int_{t_i}^{t_f} \frac{d}{dt}(m\vec{v})\,dt$$

$$= m\vec{v}\,\Big]_{t_i}^{t_f}$$

$$\boxed{\underbrace{\int_{t_i}^{t_f} \vec{F}\,dt}_{\text{Impulse of the force } \vec{F}} = \underbrace{m\vec{v}_f - m\vec{v}_i}_{\text{Change in momentum}}}$$

* Momentum is a <u>vector</u>
Impulse of a force is a <u>vector</u>

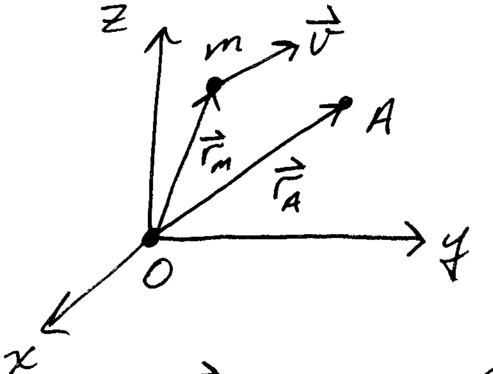
If $\Sigma F_x = 0$ then momentum in x-direction is conserved

$\Sigma F_y = 0 \rightarrow$ momentum in y is conserved

$\Sigma F_z = 0 \rightarrow$ momentum in z is conserved

For single particle systems this linear momentum principle does not tell us all that much. What tends to be more interesting and useful is angular momentum.

Angular Impulse and Momentum



$$\vec{r}_{m/A} = \vec{r}_m - \vec{r}_A$$

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{r}_{m/A} \times \vec{F} = \vec{r}_{m/A} \times m\frac{d\vec{v}}{dt}$$

Consider $\frac{d}{dt}\left(\vec{r}_{m/A} \times m\vec{v}\right)$

$$= \frac{d}{dt}\left(\vec{r}_m \times m\vec{v} - \vec{r}_A \times m\vec{v}\right)$$

$$= \frac{d\vec{r}_m}{dt} \times m\vec{v} + \vec{r}_m \times m\frac{d\vec{v}}{dt}$$

$$- \frac{d\vec{r}_A}{dt} \times m\vec{v} - \vec{r}_A \times m\frac{d\vec{v}}{dt}$$

0 b/c $\vec{v} \times \vec{v} = 0$

$$= \cancel{\vec{v} \times m\vec{v}} - \vec{v}_A \times m\vec{v} + \vec{r}_{m/A} \times m\frac{d\vec{v}}{dt}$$

$$\therefore \vec{r}_{m/A} \times m\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{r}_{m/A} \times m\vec{v}\right) + \vec{v}_A \times m\vec{v}$$

So $\underbrace{\vec{r}_{m/A} \times \vec{F}}_{\vec{M}_A} = \frac{d}{dt}\big(\underbrace{\vec{r}_{m/A} \times m\vec{v}}_{\vec{h}_A}\big) + \vec{v}_A \times m\vec{v}$

$$\therefore \vec{M}_A = \frac{d\vec{h}_A}{dt} + \vec{v}_A \times m\vec{v}$$

Now if we take A to be a fixed point, i.e. $\vec{v}_A = 0$ then

$$\vec{M}_A = \frac{d\vec{h}_A}{dt} \quad , \quad A \text{ fixed}$$

$$\int_{t_i}^{t_f} \vec{M}_A \, dt = \int_{t_i}^{t_f} \frac{d\vec{h}_A}{dt} \, dt$$

$$\underbrace{\int_{t_i}^{t_f} \vec{M}_A \, dt}_{\text{Angular impulse}} = \underbrace{\vec{h}_{A,f} - \vec{h}_{A,i}}_{\text{Change in angular momentum}} \quad * A \text{ fixed}$$

Again, this is a vector equation.

For planar problems we are usually interested in moments and changes in angular momentum in the z-direction.

Hence, if $\sum M_z^A = 0$ about the fixed point A then the angular momentum in the z-direction about point A is conserved.

MECH 211 Supplement on Angular Momentum

$$\vec{h}_A = \vec{r}_{m/A} \times m\vec{v}$$

Conservation of angular momentum is usually of interest when solving problems in rotating systems. In such systems, when the axis of rotation is fixed, point A should be taken to be on the axis of rotation.

In cylindrical coordinates we can then write

$$\vec{r}_{m/A} = R\vec{e}_R + z\vec{e}_z$$

and

$$\vec{v} = \dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z$$

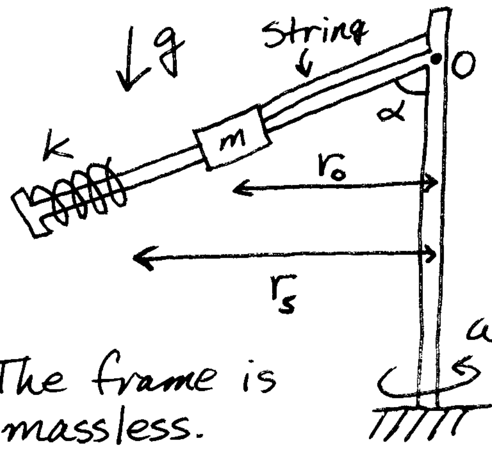
then $\vec{r}_{m/A} \times m\vec{v} = (R\vec{e}_R + z\vec{e}_z) \times m(\dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z)$

$$= m\Big[R\dot{R}\underbrace{\vec{e}_R \times \vec{e}_R}_{0} + R^2\dot{\theta}\underbrace{\vec{e}_R \times \vec{e}_\theta}_{\vec{e}_z} + R\dot{z}\underbrace{\vec{e}_R \times \vec{e}_z}_{-\vec{e}_\theta}$$
$$+ z\dot{R}\underbrace{\vec{e}_z \times \vec{e}_R}_{\vec{e}_\theta} + zR\dot{\theta}\underbrace{\vec{e}_z \times \vec{e}_\theta}_{-\vec{e}_r} + z\dot{z}\underbrace{\vec{e}_z \times \vec{e}_z}_{0}\Big]$$

$$\vec{h}_A = m\left[-zR\dot{\theta}\vec{e}_r + (z\dot{R} - R\dot{z})\vec{e}_\theta + R^2\dot{\theta}\vec{e}_z\right]$$

In most cases we are interested in angular momentum and conservation of angular momentum in the z-direction $\rightarrow h_z^A = mR^2\dot{\theta} = mRv_\theta$

Example Problem



Ⓐ Determine the tension in the string just before it breaks.

Ⓑ Determine $\vec{a}$ just after the string breaks.

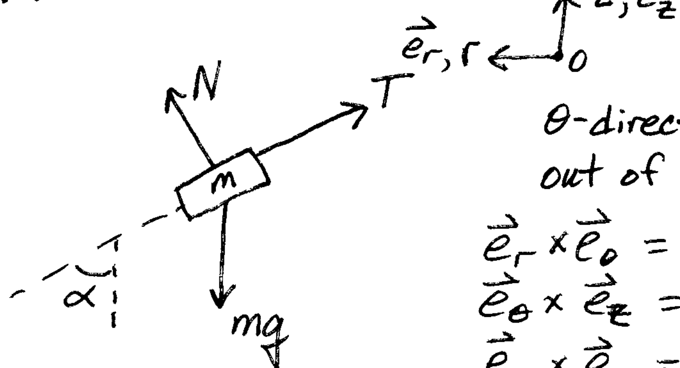
The frame is massless.

$\omega_o = \dot{\theta}$ prior to break

Rotates freely → No moment in z-direction applied at the base.

Ⓒ Determine the maximum deflection of the spring and the acceleration of the mass at that point.

Ⓐ Draw FBD



θ-direction is out of the page

$\vec{e}_r \times \vec{e}_\theta = \vec{e}_z$

$\vec{e}_\theta \times \vec{e}_z = \vec{e}_r$

$\vec{e}_z \times \vec{e}_r = \vec{e}_\theta$

* Note, since the frame is massless <u>and</u> it is rotating freely it cannot apply a force to the mass in the θ-direction.

$$\Sigma F_z = -mg + N\sin\alpha + T\cos\alpha = ma_z = m\ddot{z}$$

$$\Sigma F_r = N\cos\alpha - T\sin\alpha = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Sigma F_\theta = 0 = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Before the string breaks $r = r_0$ and $z = -r_0 \cot\alpha$, therefore $\dot{r} = 0$, $\ddot{r} = 0$, $\dot{z} = 0$, $\ddot{z} = 0$ before the string breaks.

$$\rightarrow \quad -mg + N\sin\alpha + T\cos\alpha = 0$$

$$N\cos\alpha - T\sin\alpha = -m r_0 \omega_0^2$$

$$N = T\tan\alpha + -m r_0 \omega_0^2 \frac{1}{\cos\alpha}$$

$$-mg + T\frac{\sin^2\alpha}{\cos\alpha} + -m r_0 \omega_0^2 \frac{\sin\alpha}{\cos\alpha} + T\frac{\cos^2\alpha}{\cos\alpha} = 0$$

$$\therefore \quad \boxed{T = mg\cos\alpha + m r_0 \omega_0^2 \sin\alpha}$$

(B) After string breaks $T = 0$, also we cannot have jumps in $\dot{r}$ or $\dot{z}$ or $\dot{\theta}$ because we do not have an impact. We can have jumps in $\ddot{r}$, $\ddot{z}$ and $\ddot{\theta}$.

$$\Sigma F_z = -mg + N\sin\alpha = m\ddot{z}$$

$$\Sigma F_r = N\cos\alpha = m(\ddot{r} - r\dot{\theta}^2)$$

$$\Sigma F_\theta = 0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$\rightarrow \quad \dot{r} = 0 \quad \rightarrow \quad m r_0 \ddot{\theta} = 0 \rightarrow \ddot{\theta} = 0$$

$$r = r_0$$

Note that the mass is constrained to move along the rod in such a way that

$$\tan\alpha = \frac{r}{-z} \rightarrow \quad r = -z\tan\alpha$$

$$\dot{r} = -\dot{z}\tan\alpha$$

$r = r_0$, $z = -r_0\cot\alpha$ $\qquad \ddot{r} = -\ddot{z}\tan\alpha$

$\dot{r} = 0$, $\dot{z} = 0$, $\dot{\theta} = \omega_0$

$$\Sigma F_r \rightarrow N\cos\alpha = m(\ddot{r} - r_0\omega_0^2) = m(-\ddot{z}\tan\alpha - r_0\omega_0^2)$$

$$\therefore N = -m\ddot{z}\frac{\sin\alpha}{\cos^2\alpha} - mr\omega_0^2\frac{1}{\cos\alpha}$$

then $\Sigma F_z \rightarrow -mg - m\ddot{z}\frac{\sin^2\alpha}{\cos^2\alpha} - mr\omega_0^2\frac{\sin\alpha}{\cos\alpha} = m\ddot{z}$

$$\therefore \cancel{m}\ddot{z}\left(\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}\right) = -\cancel{m}g - \cancel{m}r\omega_0^2\frac{\sin\alpha}{\cos\alpha}$$

$$\boxed{\begin{aligned} \ddot{z} &= -g\cos^2\alpha - r\omega_0^2\sin\alpha\cos\alpha \\ \rightarrow \ddot{r} &= g\sin\alpha\cos\alpha + r\omega_0^2\sin^2\alpha \end{aligned}}$$

Then, along with $\dot{r} = \dot{z} = \ddot{\theta} = 0$, we have

$$\vec{a} = (\ddot{r} - r_0\omega_0^2)\vec{e}_r + 0\,\vec{e}_\theta + \ddot{z}\,\vec{e}_z$$

$$\boxed{\begin{aligned} \vec{a} = {} & (g\sin\alpha\cos\alpha - r_0\omega_0^2\cos^2\alpha)\vec{e}_r \\ & -(g\cos^2\alpha + r\omega_0^2\sin\alpha\cos\alpha)\vec{e}_z \end{aligned} \quad \text{Just after break}}$$

c) Energy is conserved (PE in mg and $K\delta$)

0 b/c N is normal to the motion

$$KE_i + PE_i + \cancel{W^{nc}}^{\,0} = KE_f + PE_f$$

$$\tfrac{1}{2} m v_i^2 + mgh_i + \tfrac{1}{2} k \delta_i^2 = \tfrac{1}{2} m v_f^2 + mgh_f + \tfrac{1}{2} k \delta_f^2$$

$$\vec{v}_i = \dot{r}_i \vec{e}_r + r_i \dot{\theta}_i \vec{e}_\theta + \dot{z} \vec{e}_z$$

$$= 0\, \vec{e}_r + r_0 \omega_0 \vec{e}_\theta + 0\, \vec{e}_z = r_0 \omega_0 \vec{e}_\theta$$

$$\therefore v_i^2 = \vec{v}_i \cdot \vec{v}_i = r_0^2 \omega_0^2$$

$h_i = - r_0 \cot\alpha$ (Datum for h set at $z=0$)

$\delta_i = 0$ (Spring has not been compressed)

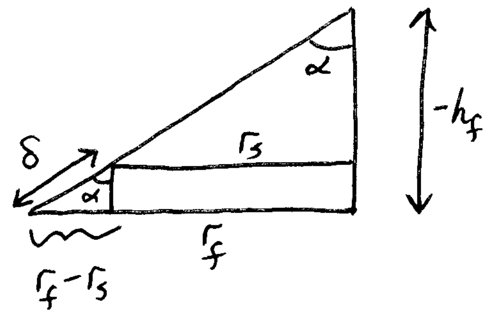
$$\therefore \boxed{KE_i + PE_i = \tfrac{1}{2} m r_0^2 \omega_0^2 - mg r_0 \cot\alpha}$$

$$\vec{v}_f = \dot{r}_f \vec{e}_r + r_f \dot{\theta}_f \vec{e}_\theta + \dot{z} \vec{e}_z$$

but at maximum compression the mass has zero velocity <u>with respect to the frame</u>, i.e. $\dot{r}_f = 0$ and $\dot{z}_f = 0$.

$$\therefore \vec{v}_f = r_f \dot{\theta}_f \vec{e}_\theta \rightarrow v_f^2 = r_f^2 \dot{\theta}_f^2$$

Finally, h_f and δ_f can be related to r_f through the geometry of the system.



$$h_f = -r_f \cot\alpha$$

$$\delta = (r_f - r_s)/\sin\alpha$$

$$\therefore KE_f + PE_f = \frac{1}{2} m r_f^2 \dot{\theta}_f^2 - mg\, r_f \cot\alpha + \frac{1}{2} k \frac{(r_f - r_s)^2}{\sin^2\alpha}$$

$$\rightarrow \boxed{\frac{1}{2} m r_0^2 \omega_0^2 - mg r_0 \cot\alpha = \frac{1}{2} m r_f^2 \dot{\theta}_f^2 - mg r_f \cot\alpha + \frac{1}{2} k \frac{(r_f - r_s)^2}{\sin^2\alpha}}$$

1 Equation but 2 unknowns r_f and $\dot{\theta}_f$

Angular momentum is conserved in the z-direction about point O because O is fixed and $\Sigma M_z^O = 0$. Note, this would not be true if for example the frame was being driven to rotate at a constant angular velocity.

$$\Sigma M_z^O = 0 \rightarrow h^O_{z,f} = h^O_{z,i}$$ ← Here h denotes angular momentum, not height

$$\vec{h}_O = \vec{r}_{m/O} \times m\vec{v}$$

$$\vec{r}_{m/O,i} = r_i \vec{e}_r + z_i \vec{e}_z = r_0 \vec{e}_r - r_0 \cot\alpha\, \vec{e}_z$$

$$\vec{v}_i = r_0 \omega_0 \vec{e}_\theta$$

$$\therefore \vec{h}_{O,i} = m r_0^2 \omega_0\, \vec{e}_r \times \vec{e}_\theta - m r_0^2 \omega_0 \cot\alpha\, \vec{e}_z \times \vec{e}_\theta$$

$$\rightarrow \vec{h}_{o,i} = \underbrace{m r_o^2 \omega_o}_{h^o_{z,i}} \vec{e}_z + m r_o^2 \omega_o \cot\alpha \, \vec{e}_r$$

Similarly : $\vec{h}_{o,f} = (r_f \vec{e}_r - r_f \cot\alpha \vec{e}_z) \times (m r_f \dot{\theta}_f \vec{e}_\theta)$

$$= \underbrace{m r_f^2 \dot{\theta}_f}_{h^o_{z,f}} \vec{e}_z + m r_f^2 \dot{\theta}_f \cot\alpha \, \vec{e}_r$$

$$\therefore \quad m r_f^2 \dot{\theta}_f = m r_o^2 \omega_o$$

$$\dot{\theta}_f = \frac{r_o^2}{r_f^2} \omega_o$$

$$\therefore \underbrace{\frac{1}{2} m r_o^2 \omega_o^2 - m g r_o \cot\alpha = \frac{1}{2} m \frac{r_o^4}{r_f^2} \omega_o^2 - m g r_f \cot\alpha + \frac{1}{2} k \frac{(r_f - r_s)^2}{\sin^2\alpha}}_{\text{Solve for } r_f.}$$

In general this will result in a ~~[illegible]~~ quartic equation.

Let's look at a specific case.

$r_o \omega_o^2 \approx$ acceleration $\longrightarrow$ take $r_o \omega_o^2 = g$

$k \approx \frac{\text{Force}}{\text{length}}$ $\longrightarrow$ take $k = \frac{mg}{r_o}$

take $r_s = 2 r_o$

take $\alpha = 45^\circ$

Then: $\frac{1}{2} m\cancel{g}r_0 - m\cancel{g}r_0 = \frac{1}{2} m\cancel{g}r_0 \left(\frac{r_0}{r_f}\right)^2 - m\cancel{g}r_0\left(\frac{r_f}{r_0}\right) + m\cancel{g}r_0\left[\left(\frac{r_f}{r_0}\right) - 2\right]^2$

$$-\frac{1}{2} = \frac{1}{2}\left(\frac{r_0}{r_f}\right)^2 - \left(\frac{r_f}{r_0}\right) + \left(\frac{r_f}{r_0}\right)^2 - 4\left(\frac{r_f}{r_0}\right) + 4$$

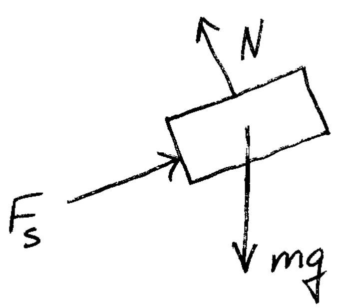
$$\left(\frac{r_f}{r_0}\right)^4 - 5\left(\frac{r_f}{r_0}\right)^3 + \frac{9}{2}\left(\frac{r_f}{r_0}\right)^2 + \frac{1}{2} = 0$$

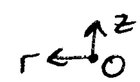
$$\left(\frac{r_f}{r_0}\right) = 1.29506,\ \boxed{3.80979}\quad \left(\begin{array}{c}\text{other 2 roots}\\ \text{are imaginary}\end{array}\right)$$

Physically realistic root

This root is between r_0 and r_s

What is $\vec{a}$ at this point? Draw FBD.





$$\Sigma F_z = -mg + N\sin\alpha + F_s\cos\alpha = m\ddot{z}$$

$$\Sigma F_r = N\cos\alpha - F_s\sin\alpha = m(\ddot{r} - r\dot{\theta}^2)$$

Recall: $\ddot{r} = -\ddot{z}\tan\alpha$

$F_s = k\delta = k(r_f - r_s)/\sin\alpha \leftarrow$ known

$$\therefore -mg + N\sin\alpha + k(r_f - r_s)\frac{\cos\alpha}{\sin\alpha} = m\ddot{z}$$

$$N\cos\alpha - k(r_f - r_s) = m\left(-\ddot{z}\frac{\sin\alpha}{\cos\alpha} - r_f\dot{\theta}_f^2\right)$$

$$N = k(r_f - r_s)/\cos\alpha - m\ddot{z}\frac{\sin\alpha}{\cos^3\alpha} - m r_f \dot{\theta}_f^2 / \cos\alpha$$

$$-mg + k(r_f - r_s)\frac{\sin\alpha}{\cos\alpha} - m\ddot{z}\frac{\sin^2\alpha}{\cos^2\alpha} - m r_f \dot{\theta}_f^2 \frac{\sin\alpha}{\cos\alpha} + k(r_f - r_s)\frac{\cos\alpha}{\sin\alpha} = m\ddot{z}$$

$$m\ddot{z}\left(\frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha}\right) = -mg + k(r_f - r_s)\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right) - m r_f \dot{\theta}_f^2 \frac{\sin\alpha}{\cos\alpha}$$

$$\ddot{z} = -g\cos^2\alpha + \frac{k}{m}(r_f - r_s)\left(\sin\alpha\cos\alpha + \underbrace{\frac{\cos^3\alpha}{\sin\alpha}}_{\cos\alpha(1-\sin^2\alpha)/\sin\alpha}\right) - r_f \dot{\theta}_f^2 \sin\alpha\cos\alpha$$

$$\frac{\cos\alpha}{\sin\alpha} - \frac{\cos\alpha\sin^2\alpha}{\sin\alpha}$$

$$\ddot{r} = -\ddot{z}\tan\alpha$$

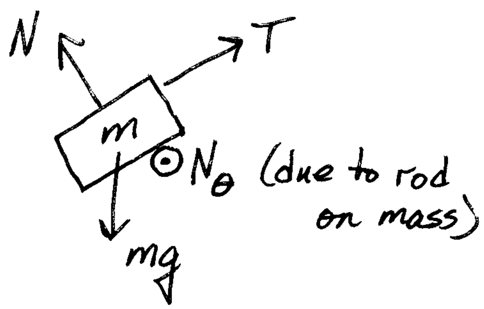
$$\vec{a} = (\ddot{r} - r_f\dot{\theta}_f^2)\vec{e}_r + (r_f\underbrace{\ddot{\theta}}_{0 \text{ b/c } \Sigma F_\theta = 0} + 2\underbrace{\dot{r}}_{0}\dot{\theta}_f)\vec{e}_\theta + \ddot{z}\vec{e}_z$$

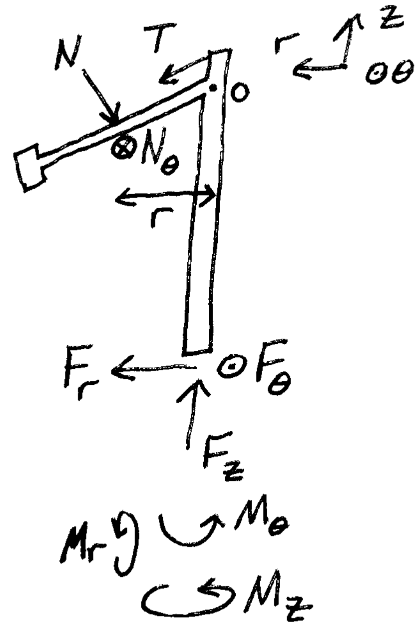
$$\ddot{z} = -g\cos^2\alpha + \frac{k}{m}(r_f - r_s)\frac{\cos\alpha}{\sin\alpha} - r_f\dot{\theta}_f^2 \sin\alpha\cos\alpha$$

$$\ddot{r} = g\sin\alpha\cos\alpha + -\frac{k}{m}(r_f - r_s) + r_f\dot{\theta}_f^2 \sin^2\alpha$$

$$\boxed{\begin{aligned}\vec{a} &= \left[g\sin\alpha\cos\alpha - \frac{k}{m}(r_f - r_s) - r_f\dot{\theta}_f^2\cos^2\alpha\right]\vec{e}_r \\ &\quad - \left[g\cos^2\alpha - \frac{k}{m}(r_f - r_s)\frac{\cos\alpha}{\sin\alpha} + r_f\dot{\theta}_f^2\sin\alpha\cos\alpha\right]\vec{e}_z\end{aligned}}$$

Why could we claim that the normal force that the rod places on the mass in the θ-direction was zero? Consider these 2 FBD's





We will analyze the dynamics of rigid bodies later, but note here that if a rigid body is <u>massless</u> then the equations of motion for the body, i.e. Newton's laws of motion, reduce to the equations of statics, i.e.

$$\Sigma \vec{F} = 0 \quad \text{and} \quad \Sigma \vec{M}_A = 0 \quad \text{for a massless rigid body.}$$

Otherwise the angular or linear accelerations of the rigid body would go to infinity.

Back to our frame.

Consider the ΣM_z^o for the frame.

$$\Sigma M_z^o = M_z + r N_\theta = 0 \leftarrow \text{b/c massless}$$

$$\therefore \quad N_\theta = M_z / r$$

But in our problem we said that the frame was <u>rotating freely</u> and that means that $M_z = 0$.

$$\therefore \text{ Since } M_z = 0 \; ; \rightarrow N_\theta = 0$$

However, if the frame was being driven at some constant angular velocity (or otherwise specified angular velocity) then M_z would generally not be equal to zero and then N_θ would not be equal to zero either.

So to recap: $N_\theta = 0$ in our problem because the frame was massless <u>and</u> it was rotating freely about its base in the z-direction.