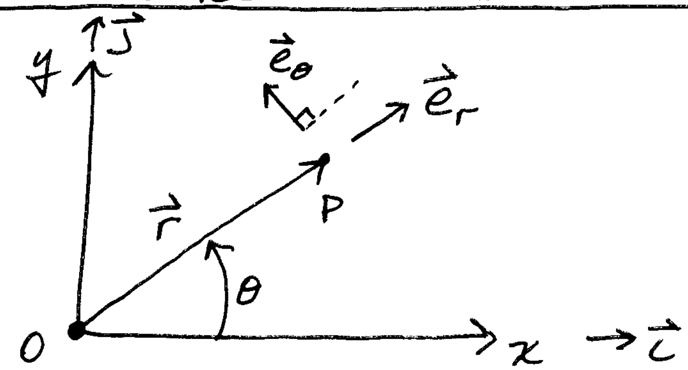
Kinematics in Polar Coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r = |\vec{r}|$$

$$\vec{i} = \vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta$$
$$\vec{j} = \vec{e}_r\sin\theta + \vec{e}_\theta\cos\theta$$

Similarly $\quad \vec{e}_r = \vec{i}\cos\theta + \vec{j}\sin\theta$

$$\vec{e}_\theta = -\vec{i}\sin\theta + \vec{j}\cos\theta$$

Recall $\quad \vec{r} = x\vec{i} + y\vec{j}$

$$= (r\cos\theta)(\vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta) + (r\sin\theta)(\vec{e}_r\sin\theta + \vec{e}_\theta\cos\theta)$$

$$= r\cos^2\theta\,\vec{e}_r - r\cos\theta\sin\theta\,\vec{e}_\theta + r\sin^2\theta\,\vec{e}_r + r\sin\theta\cos\theta\,\vec{e}_\theta$$

$$= r(\cos^2\theta + \sin^2\theta)\vec{e}_r = r\vec{e}_r$$

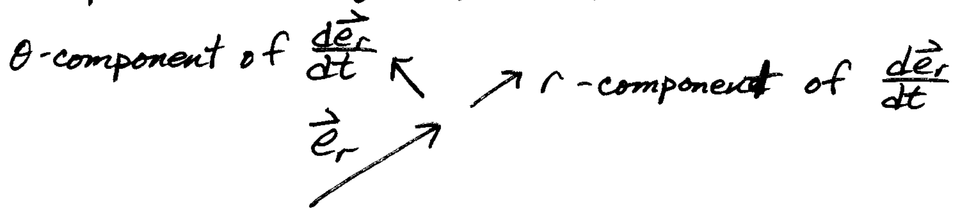
Note: $\vec{e}_r$ is the unit vector in the r direction so as we should expect

$$\boxed{\vec{r} = r\vec{e}_r}$$

Now: $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(r\,\vec{e}_r)$

$$= \underbrace{\frac{dr}{dt}}_{\dot{r}}\vec{e}_r + r\underbrace{\frac{d\vec{e}_r}{dt}}_{?}$$

Recall: Any vector, including $\frac{d\vec{e}_r}{dt}$, can be decomposed into orthogonal components. So we can take components of $\frac{d\vec{e}_r}{dt}$ in the $\vec{e}_r$ and $\vec{e}_\theta$ directions.



But $\vec{e}_r$ is a unit vector so its length cannot change. Therefore the r-component of $\frac{d\vec{e}_r}{dt}$ must be zero. In general the rate of change of any unit vector will always be perpendicular to itself (or zero as in the case of $\vec{\imath}, \vec{\jmath}, \vec{k}$).

$$\frac{d}{dt}(\vec{e}_r) = \frac{d}{dt}(\vec{\imath}\cos\theta + \vec{\jmath}\sin\theta)$$

$$= \cancel{\frac{d\vec{\imath}}{dt}}^{0}\cos\theta + \vec{\imath}\frac{d}{dt}(\cos\theta) + \cancel{\frac{d\vec{\jmath}}{dt}}^{0}\sin\theta + \vec{\jmath}\frac{d}{dt}(\sin\theta)$$

$$= \vec{\imath}\underbrace{(-\sin\theta)}_{\frac{d}{d\theta}(\cos\theta)}\frac{d\theta}{dt} + \vec{\jmath}\underbrace{(\cos\theta)}_{\frac{d}{d\theta}(\sin\theta)}\frac{d\theta}{dt}$$

$$\therefore \frac{d\vec{e}_r}{dt} = \dot{\theta}\underbrace{\left(-\vec{\imath}\sin\theta + \vec{\jmath}\cos\theta\right)}_{\vec{e}_\theta}$$

$$\boxed{\frac{d\vec{e}_r}{dt} = \dot{\theta}\,\vec{e}_\theta}$$

Similarly: $\frac{d\vec{e}_\theta}{dt} = \frac{d}{dt}\left(-\vec{\imath}\sin\theta + \vec{\jmath}\cos\theta\right)$

$$= -\vec{\imath}\,(\cos\theta)\,\dot{\theta} + \vec{\jmath}\,(-\sin\theta)\,\dot{\theta}$$

$$= -\dot{\theta}\underbrace{\left(\vec{\imath}\cos\theta + \vec{\jmath}\sin\theta\right)}_{\vec{e}_r}$$

$$\boxed{\frac{d\vec{e}_\theta}{dt} = -\dot{\theta}\,\vec{e}_r}$$

Back to $\vec{v}$. $\therefore$ $\boxed{\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta}$

$\dot{r}$ is the normal or radial component o $\vec{v}$ and $r\dot{\theta}$ is the tangential or θ component of $\vec{v}$, i.e.

$$v_n = v_r = \dot{r} \quad \text{and}$$

$$v_t = v_\theta = r\dot{\theta}$$

[redacted] Now for $\vec{a}$.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta\right)$$

$$= \underbrace{\frac{d\dot{r}}{dt}}_{\ddot{r}}\vec{e}_r + \dot{r}\underbrace{\frac{d\vec{e}_r}{dt}}_{\dot{\theta}\vec{e}_\theta} + \underbrace{\frac{dr}{dt}}_{\dot{r}}\dot{\theta}\vec{e}_\theta + r\underbrace{\frac{d\dot{\theta}}{dt}}_{\ddot{\theta}}\vec{e}_\theta + r\dot{\theta}\underbrace{\frac{d\vec{e}_\theta}{dt}}_{-\dot{\theta}\vec{e}_r}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta}$$

radial / normal acceleration $a_r = a_n = \ddot{r} - r\dot{\theta}^2$

$\ddot{r} \rightarrow$ accelerating along a ray

$r\dot{\theta}^2 = \frac{(r\dot{\theta})^2}{r} = \frac{v_t^{\,2}}{r} \rightarrow$ centripetal acceleration

tangential / θ acceleration $a_t = a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$r\ddot{\theta} \rightarrow$ accelerating around a circle

$2\dot{r}\dot{\theta} \rightarrow$ unintuitive "Coriolis acceleration"

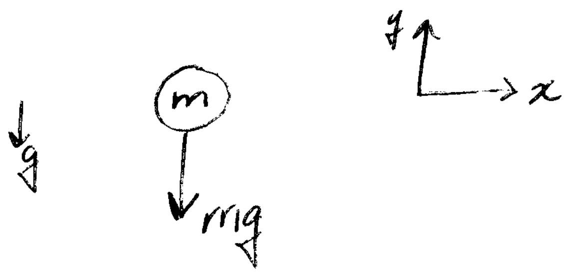
Kinetics

$\Sigma \vec{F} = m\vec{a}$ $\rightarrow$ leads to equations of motion that are in many cases described by differential equations.

Projectiles

Projectiles act only under the force of gravity (if we neglect wind resistance).

FBD



$$\Sigma F_x = 0 = m a_x$$

$$\Sigma F_y = -mg = m a_y$$

$$x \rightarrow \quad a_x = 0 \rightarrow v_x = c_1 \rightarrow x = c_1 t + c_2$$

Apply initial conditions : $x(t=0) = x_0$

$$v_x(t=0) = \dot{x}(t=0) = v_{x0}$$

$$\therefore \quad c_1 = v_{x0} \quad \text{and} \quad c_2 = x_0$$

$$x = x_0 + v_{x0} t$$, i.e. constant velocity motion in x-direction

$y \rightarrow \quad a_y = -g \rightarrow v_y = -gt + c_3 \rightarrow y = -\frac{1}{2}gt^2 + c_3 t + c_4$

Apply initial conditions: $y(t=0) = y_0$

$$v_y(t=0) = \dot{y}(t=0) = v_{y0}$$

$\therefore \quad c_3 = v_{y0} \quad$ and $\quad c_4 = y_0$

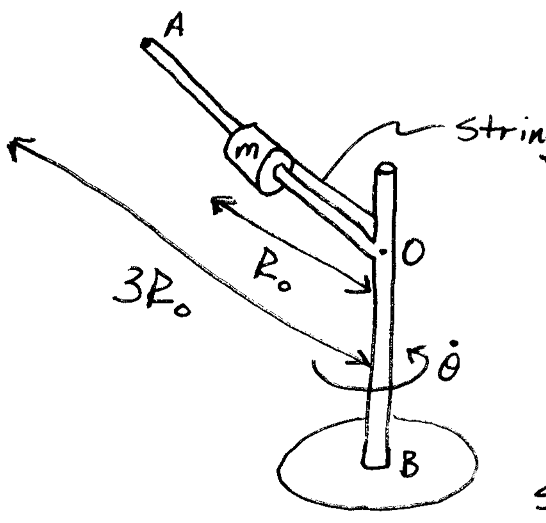
$$\therefore \quad y = y_0 + v_{y0} t + -\frac{1}{2} g t^2$$

Hence the position as a function of time for a projectile can be given as

$$\boxed{\begin{aligned} x &= x_0 + v_{x0} t \\ y &= y_0 + v_{y0} t - \frac{1}{2} g t^2 \end{aligned}}$$

Projectile motion with gravity in $-y$ direction.

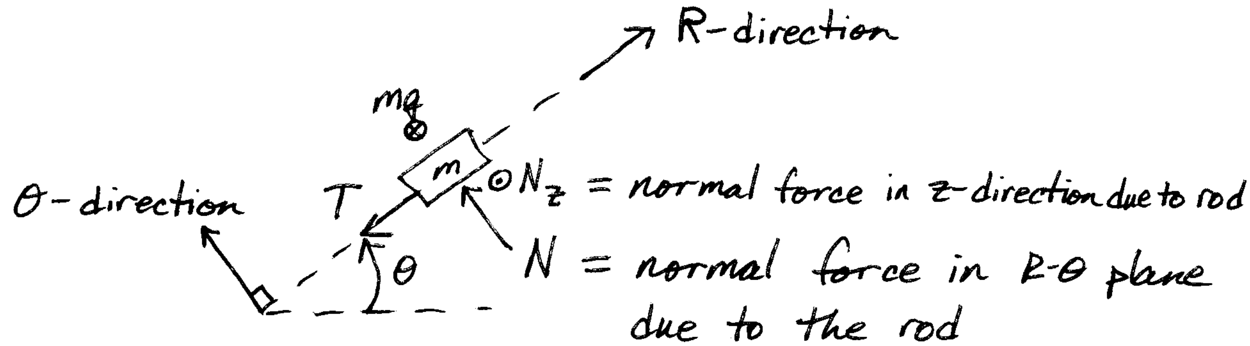
Example Problem 13.59



$\dot{\theta} = \omega =$ constant frictionless slider of mass m. String cut at time $t=0$. Determine T in string before cut, $\ddot{R}$ after cut, equation of motion for $R(t)$, speed when slider reaches A.

We will be using $\Sigma\vec{F} = m\vec{a}$. Any time you see $\Sigma\vec{F}$ you should automatically start drawing a FBD.

Looking down on the rod OB



$$\Sigma F_z = ma_z \rightarrow N_z - mg = m\cancel{a_z}^{\,0} \therefore N_z = mg$$

$$\Sigma F_\theta = ma_\theta \rightarrow N = m(R\cancel{\ddot{\theta}}^{\,0} + 2\dot{R}\underbrace{\dot{\theta}}_{\omega})$$

$$\therefore N = 2m\dot{R}\omega \qquad \dot{R} = ?$$

$$\Sigma F_R = ma_R \rightarrow -T = m(\ddot{R} - R\dot{\theta}^2)$$

Just prior to cutting the string $R = R_o$, $\dot{R} = 0$, $\ddot{R} = 0$

$$\therefore N = 2m\dot{R}\omega = 0 \text{ and } -T = m(0 - R_o\omega^2)$$

$$\therefore T = mR_o\omega^2 \text{ before cut}$$

Just after the cut $\quad T = 0$
$\qquad\qquad\qquad\qquad R = R_0$
$\qquad\qquad\qquad\qquad \dot{R} = 0$

Note that when we do not have impacts, velocity components cannot change instantaneously. However, acceleration components <u>can</u>.

$$\left.\begin{array}{l} \therefore \quad N = 2m \cdot 0 \cdot \omega = 0 \\ 0 = m(\ddot{R} - R_0\omega^2) \rightarrow \ddot{R} = R_0\omega^2 \end{array}\right\} \text{Just after cut.}$$

The equation of motion after the string has been cut is from $\Sigma F_R = m a_R$, i.e.

$$-\underbrace{T = 0}_{\text{after cut}} = m(\ddot{R} - R\dot{\theta}^2)$$

$$\therefore \quad \ddot{R} - \omega^2 R = 0 \quad \text{or} \quad \boxed{\frac{d^2R}{dt^2} - \omega^2 R = 0}$$

We were asked for the speed at A.

Recall $\quad \vec{v} = \dot{R}\,\vec{e}_r + R\dot{\theta}\,\vec{e}_\theta$

at $A \qquad R = 3R_0 \;,\; \dot{\theta} = \omega \;,\; \dot{R} = ?$

We will go through 2 methods to get $\dot{R}$.

<u>Method 1</u> : Solve the ODE

$$\frac{d^2R}{dt^2} - \omega^2 R = 0 \rightarrow \text{2nd order, linear ODE with constant coefficients}$$

Assume a solution of the form $R(t) = A\exp[\alpha t]$

$$\frac{dR}{dt} = A\alpha \exp[\alpha t]$$

$$\frac{d^2R}{dt^2} = A\alpha^2 \exp[\alpha t]$$

Plug into our equation: $\cancel{A}\alpha^2 \cancel{\exp[\alpha t]} - \cancel{A}\omega^2 \cancel{\exp[\alpha t]} = 0$

$$\therefore -\omega^2 + \alpha^2 = 0$$

$$\alpha = \pm\omega$$

Then the general solution is $R(t) = A\exp[\omega t] + B\exp[-\omega t]$

$$\dot{R}(t) = A\omega\exp[\omega t] - B\omega\exp[\omega t]$$

Apply initial conditions to determine A & B.

$$R(t=0) = R_0 \quad \text{and} \quad \dot{R}(t=0) = 0$$

$$\therefore A + B = R_0 \qquad \omega(A-B) = 0$$

$$A = B = \frac{R_0}{2}$$

So $R(t) = \frac{R_0}{2}\left[\exp(\omega t) + \exp(-\omega t)\right] = R_0\cosh(\omega t)$

at A, $R = 3R_0 \rightarrow R(t_A) = R_0 \cosh(\omega t_A) = 3R_0$

$$\therefore \omega t_A = \text{arccosh}(3) = 1.763$$

$$\therefore \dot{R}(t_A) = R_0 \omega \sinh(\omega t_A) = 2.828 R_0 \omega$$

$$\therefore \vec{v}_A = 2.828 R_0 \omega \vec{e}_r + 3 R_0 \omega \vec{e}_\theta$$

$$\boxed{v_A = R_0\omega\sqrt{2.828^2 + 3^2} = 4.123 R_0 \omega}$$

Method 2: Integration Tricks

Recall $\ddot{R} = \omega^2 R$ from our equation of motion

but $\ddot{R} = \dfrac{d\dot{R}}{dt} = \dfrac{d\dot{R}}{dR}\dfrac{dR}{dt} = \dot{R}\dfrac{d\dot{R}}{dR}$

Combine these two equations to get

$$\dot{R}\frac{d\dot{R}}{dR} = \omega^2 R \quad \left(\text{Similar to } v\frac{dv}{ds} = f(s)\right)$$

Integrate $\rightarrow$ $\displaystyle\int_0^{\dot{R}_A} \dot{R}\, d\dot{R} = \int_{R_0}^{3R_0 = R_A} \omega^2 R\, dR$

Initial condition

$$\therefore \frac{1}{2}\dot{R}^2\Big]_0^{\dot{R}_A} = \frac{1}{2}\omega^2 R^2\Big]_{R_0}^{3R_0}$$

$$\frac{1}{2}\dot{R}_A^2 = \frac{1}{2}\omega^2\left(9R_o^2 - R_o^2\right)$$

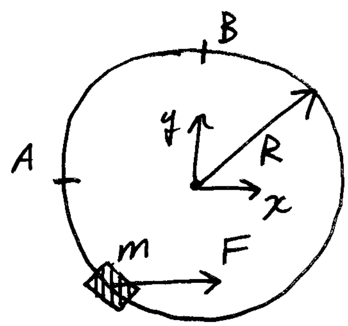
$$\therefore\ \dot{R}_A = \sqrt{8}\ \omega R_o = 2.828\,\omega R_o$$

then $\vec{v}_A = \dot{R}_A\,\vec{e}_r + R_A\omega\,\vec{e}_\theta$

$$= 2.828\,\omega R_o\,\vec{e}_r + 3R_o\omega\,\vec{e}_\theta$$

$$\therefore\ \boxed{v_A = \sqrt{\sqrt{8}^2 + 3^2}\ \omega R_o = \sqrt{17}\,\omega R_o = 4.123\,\omega R_o}$$

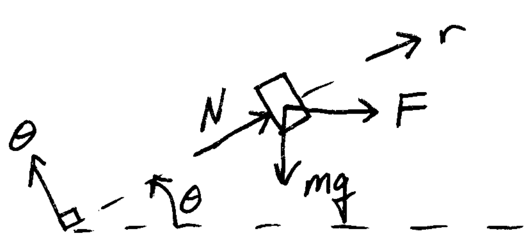
<u>Example Problem 13.55</u>



m starts from rest at A.
What is the smallest F that can cause the slider to reach B ?
F always acts in positive x.

$\Sigma\vec{F} = m\vec{a} \rightarrow$ Draw FBD

Let's draw our FBD at a positive θ angle.



$$\Sigma F_r = ma_r \rightarrow N + F\cos\theta - mg\sin\theta = m(\ddot{R} - R\dot{\theta}^2)$$

$$\Sigma F_\theta = ma_\theta \rightarrow -F\sin\theta - mg\cos\theta = m(R\ddot{\theta} + 2\dot{R}\dot{\theta})$$

but $R = \text{constant} \Rightarrow \dot{R} = 0$ and $\ddot{R} = 0$

$$\therefore \quad N + F\cos\theta - mg\sin\theta = -mR\dot{\theta}^2 \quad (r)$$

and

$$-F\sin\theta - mg\cos\theta = mR\ddot{\theta} \quad (\theta)$$

We want to find an equation that looks like $\ddot{\theta} = f(\theta)$ because if we do then we can use our integration trick that tells us $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = f(\theta)$

then $$\int_{\dot{\theta}_i}^{\dot{\theta}_f} \dot{\theta}\, d\dot{\theta} = \int_{\theta_i}^{\theta_f} f(\theta)\, d\theta$$

Well, the θ equation looks like this, i.e.

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = -\frac{F}{mR}\sin\theta - \frac{g}{R}\cos\theta$$

$\therefore$ with $\dot{\theta}_i = \dot{\theta}_A = 0$ (starts from rest)

$\theta_A = -\pi$

$\dot{\theta}_f = \dot{\theta}_B = 0$ (just reaches B)

$\theta_B = \frac{\pi}{2}$

We then can set up our integral as

$$\int_0^0 \dot{\theta}\, d\dot{\theta} = \int_{-\pi}^{\pi/2} -\frac{F}{mR}\sin\theta - \frac{g}{R}\cos\theta \, d\theta$$

$$0 = +\frac{F}{mR}\cos\theta - \frac{g}{R}\sin\theta \Big]_{-\pi}^{\pi/2}$$

$$= \frac{F}{mR}(0 - {-1}) - \frac{g}{R}(1-0)$$

$$\therefore \; +\frac{F}{mR} - \frac{g}{R} = 0$$

$$\boxed{F = mg}$$

Hence, the smallest constant horizontal force F that will allow the slider to reach B is $F = mg$.