Calculus Basics

In dynamics we are interested in functions of time and their first and second derivatives.

$$\left.\begin{aligned} f &= f(t) \\ c &= \text{constant} \end{aligned}\right\} \quad \begin{aligned} \frac{d}{dt}(cf) &= c\frac{df}{dt} \\ \frac{d^2}{dt^2}(cf) &= c\frac{d^2f}{dt^2} \end{aligned}$$

$$\left.\begin{aligned} f &= f(t) \\ g &= g(t) \end{aligned}\right\} \quad \frac{d}{dt}(fg) = g\frac{df}{dt} + f\frac{dg}{dt}$$

↑ product rule

$$\begin{aligned} \underbrace{\frac{d^2}{dt^2}(fg)} &= \frac{d}{dt}\left(g\frac{df}{dt} + f\frac{dg}{dt}\right) \\ &= \frac{dg}{dt}\frac{df}{dt} + g\frac{d^2f}{dt^2} + \frac{df}{dt}\frac{dg}{dt} + f\frac{d^2g}{dt^2} \\ &= g\frac{d^2f}{dt^2} + 2\frac{dg}{dt}\frac{df}{dt} + f\frac{d^2g}{dt^2} \end{aligned}$$

$$\left.\begin{aligned} f &= f(g) \\ g &= g(t) \end{aligned}\right\} \quad \frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

← chain rule

$$\begin{aligned} \underbrace{\frac{d^2f}{dt^2}} &= \frac{d}{dt}\left(\frac{df}{dg}\right)\frac{dg}{dt} + \frac{df}{dg}\frac{d}{dt}\left(\frac{dg}{dt}\right) \\ &= \frac{dg}{dt}\frac{d}{dg}\left(\frac{df}{dg}\right)\frac{dg}{dt} + \frac{df}{dg}\frac{d^2g}{dt^2} \\ &= \frac{d^2f}{dg^2}\left(\frac{dg}{dt}\right)^2 + \frac{df}{dg}\frac{d^2g}{dt^2} \end{aligned}$$

Dynamics

Most everything we see in everyday life can be described with Newton's law's of motion. Let's take a closer look at

$$\vec{F} = m\vec{a}$$

First, this is a vector equation. We have already spent several weeks studying what is meant by $\vec{F}$.

m is the mass of the body in consideration.

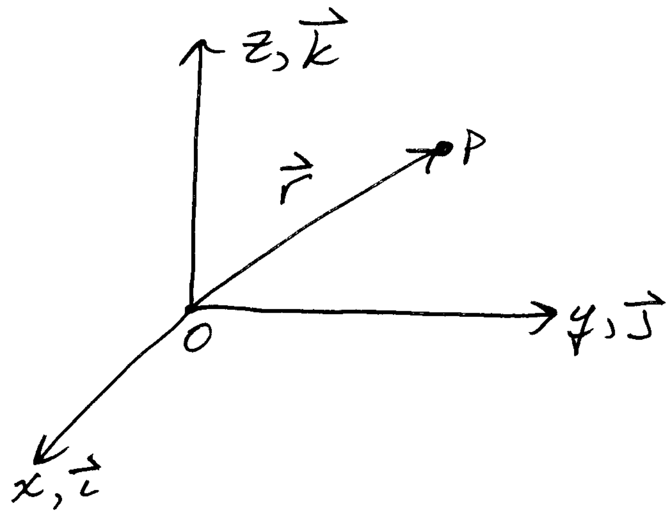
How about $\vec{a}$? The acceleration $\vec{a}$ is the rate of change of the velocity $\vec{v}$, and the velocity $\vec{v}$ is the rate of change of the position $\vec{r}$ <u>with respect to an inertial frame of reference.</u>

An inertial frame of reference is one that is fixed or moves at a constant velocity.

The physical processes described by Newton's laws of motion can be analyzed in any inertial frame of reference.

Note, a rotating reference frame is not inertial.

Consider the inertial reference frame with origin at O.



$\vec{r}$ is the position vector of particle P with respect to the origin O

$\vec{v} = \dfrac{d\vec{r}}{dt}$ = the velocity of the particle P with respect to O

Recall that a vector consists of both a magnitude and a direction. Hence, a vector changes when either its magnitude changes or its direction changes.

i.e. $\vec{A} = A\vec{e}_A \rightarrow \dfrac{d\vec{A}}{dt} = \dfrac{dA}{dt}\vec{e}_A + A\dfrac{d\vec{e}_A}{dt}$

change in magnitude

change in direction

Now, let's describe $\vec{r}$ in Cartesian coordinates.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

then
$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + x\frac{d\vec{i}}{dt} + \frac{dy}{dt}\vec{j} + y\frac{d\vec{j}}{dt} + \frac{dz}{dt}\vec{k} + z\frac{d\vec{k}}{dt}$$

however, as long as the coordinate system attached at O is not rotating, $\frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = 0$ because $\vec{i}$, $\vec{j}$ and $\vec{k}$ are always unit vectors so their magnitude can never change, and since the coordinate system is inertial (this implies non-rotating) the direction of $\vec{i}$, $\vec{j}$ and $\vec{k}$ does not change.

$$\therefore \quad \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

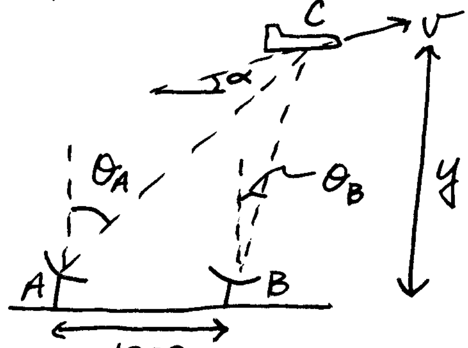
$$\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

then
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

$$\vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

Example Problem 12.25

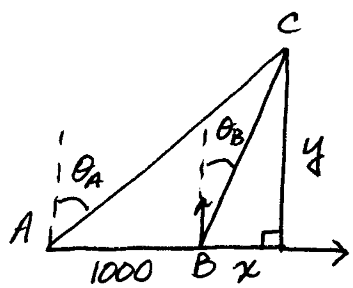


ABC lies in the plane of the paper.

$\theta_A = 30°$, $\theta_B = 22°$

$\dot{\theta}_A = 0.026 \frac{rad}{s}$, $\dot{\theta}_B = 0.032 \frac{rad}{s}$

Determine the altitude y, the speed v, and the angle of climb α at this moment.



Set origin at B such that the planes current position is at (x, y).

$$\tan\theta_A = \frac{1000 + x}{y} \qquad \tan\theta_B = \frac{x}{y}$$

$$\therefore \; x = y\tan\theta_A - 1000 \rightarrow y\tan\theta_B = y\tan\theta_A - 1000$$

$$\therefore \; \boxed{y = \frac{1000}{\tan\theta_A - \tan\theta_B} = 5770\text{ m}}$$

But the velocity of the plane is

$$\vec{v} = \dot{x}\,\vec{i} + \dot{y}\,\vec{j}$$

How do we determine $\dot{x}$ and $\dot{y}$?

We have equations for x and y, all we need to do is differentiate them.
In many cases it is easier to differentiate equations appearing early in the analysis.

$$\frac{d}{dt}\left(\tan\theta_A = \frac{1000+x}{y}\right)$$

$$\Rightarrow \frac{d}{dt}\left[y\sin\theta_A = (1000+x)\cos\theta_A\right]$$

$$\Rightarrow \frac{dy}{dt}\sin\theta_A + y\cos\theta_A \underbrace{\dot{\theta}_A}_{\frac{d\theta_A}{dt}} = \frac{dx}{dt}\cos\theta_A - (1000+x)\sin\theta_A\frac{d\theta_A}{dt}$$

Recall: $\frac{d}{dt}(\sin\theta_A) = \underbrace{\frac{d}{d\theta}(\sin\theta_A)}_{\cos\theta_A}\frac{d\theta_A}{dt}$

$$\therefore\ \dot{y}\sin\theta_A + y\dot{\theta}_A\cos\theta_A = \dot{x}\cos\theta_A - (1000+x)\dot{\theta}_A\sin\theta_A$$

2 unknowns $\dot{y}$ and $\dot{x}$, we know everything else.

Differentiate the other equation

$$\frac{d}{dt}\left(\tan\theta_B = \frac{x}{y}\right) \Rightarrow \frac{d}{dt}\left(y\sin\theta_B = x\cos\theta_B\right)$$

$$\dot{y}\sin\theta_B + y\dot{\theta}_B\cos\theta_B = \dot{x}\cos\theta_B - x\dot{\theta}_B\sin\theta_B$$

$$\therefore\ \dot{x} = \dot{y}\tan\theta_B + y\dot{\theta}_B + x\dot{\theta}_B\tan\theta_B$$

$$\rightarrow \dot{y}\sin\theta_A + y\dot{\theta}_A\cos\theta_A = (\dot{y}\tan\theta_B + y\dot{\theta}_B + x\dot{\theta}_B\tan\theta_B)\cos\theta_A$$

$$- (1000+x)\dot{\theta}_A\sin\theta_A$$

$$\therefore\ \dot{y} = \frac{(y\dot{\theta}_B + x\dot{\theta}_B\tan\theta_B)\cos\theta_A - (1000+x)\dot{\theta}_A\sin\theta_A - y\dot{\theta}_A\cos\theta_A}{\sin\theta_A - \tan\theta_B\cos\theta_A}$$

recall $x = y\tan\theta_B = 2331\ m$

$\dot{y} = 85.1\ \frac{m}{s} \rightarrow \dot{x} = 249\ \frac{m}{s}$

$$\boxed{\begin{aligned} v &= \sqrt{\dot{x}^2+\dot{y}^2} = 263\ \tfrac{m}{s} \\ \alpha &= \arctan\frac{\dot{y}}{\dot{x}} = 18.9^\circ \end{aligned}}$$

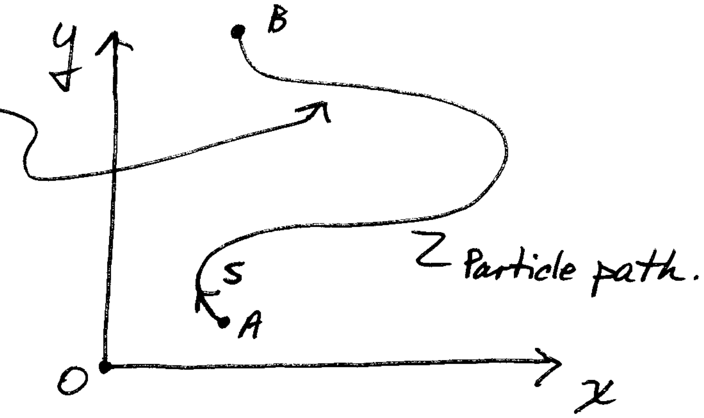
Check with $x = y\tan\theta_A - 1000$

$$\rightarrow \dot{x} = \dot{y}\tan\theta_A + y\frac{1}{\cos^2\theta_A}\dot{\theta}_A = 249\ \frac{m}{s}\ \checkmark$$

Path Coordinates

There are certain situations where we might know what the path of a particle will be a priori. For a 2-D problem this implies that the x and y coordinates of the path can be parameterized by the distance along the path s. In other words,

$$\left.\begin{aligned} x &= x(s) \\ y &= y(s) \end{aligned}\right\}$$



If we are given $s = s(t)$ then we could find the velocity and acceleration of the particle if we know $x(s)$ and $y(s)$. i.e.

$$\vec{v} = \dot{x}\,\vec{i} + \dot{y}\,\vec{j} = \frac{dx}{ds}\dot{s}\,\vec{i} + \frac{dy}{ds}\dot{s}\,\vec{j}$$

Note: $\dot{x} = \dfrac{dx}{dt} = \dfrac{dx}{ds}\dfrac{ds}{dt} = \dfrac{dx}{ds}\dot{s}$

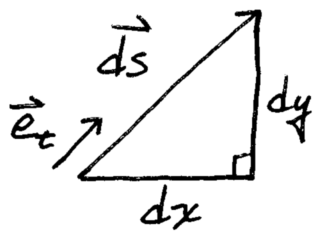
The acceleration is a bit more hairy,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\frac{dx}{ds}\dot{s}\,\vec{i} + \frac{dy}{ds}\dot{s}\,\vec{j}\right]$$

$$= \left[\frac{d}{ds}\left(\frac{dx}{dt}\right)\dot{s} + \frac{dx}{ds}\ddot{s}\right]\vec{i} + \left[\frac{d}{ds}\left(\frac{dy}{dt}\right)\dot{s} + \frac{dy}{ds}\ddot{s}\right]\vec{j}$$

$$\rightarrow \vec{a} = \left(\frac{d^2x}{ds^2}\dot{s}^2 + \frac{dx}{ds}\ddot{s} \right)\vec{i} + \left(\frac{d^2y}{ds^2}\dot{s}^2 + \frac{dy}{ds}\ddot{s} \right)\vec{j}$$

Let's look at what $\frac{dx}{ds}$ and $\frac{dy}{ds}$ are.



$$\vec{ds} = dx\,\vec{i} + dy\,\vec{j}$$

$$\vec{e}_t = \frac{\vec{ds}}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}$$

$\vec{e}_t$ is a unit vector in the s-direction.

Now let's return to $\vec{v}$.

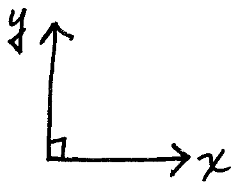
$$\boxed{\vec{v} = \dot{s}\left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} \right) = \dot{s}\,\vec{e}_t = v\,\vec{e}_t} \;**$$

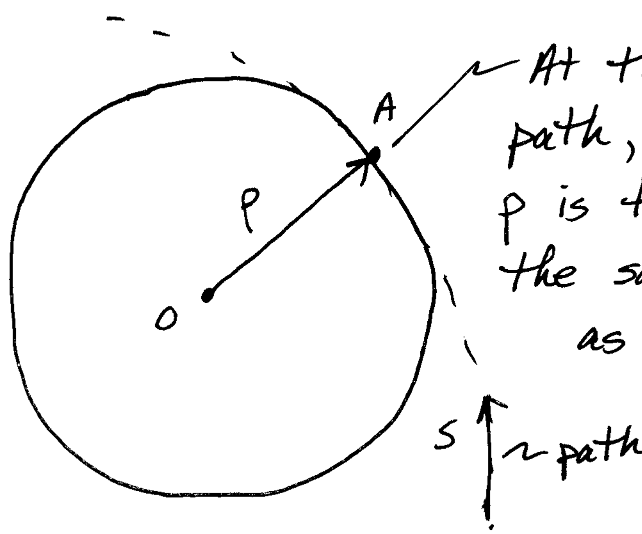
The velocity vector is alway parallel to the path.

Similarly, part of the acceleration vector is parallel to the path, i.e.

$$\vec{a} = \ddot{s}\underbrace{\left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} \right)}_{\vec{e}_t} + \dot{s}^2\underbrace{\left(\frac{d^2x}{ds^2}\vec{i} + \frac{d^2y}{ds^2}\vec{j} \right)}_{\text{What is this?}}$$

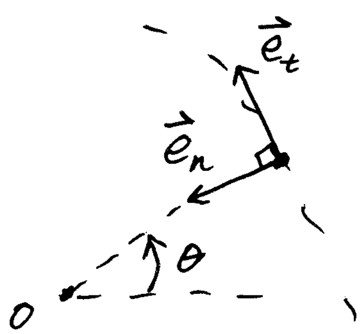
Let's look very closely at a point on the path, not so close that it looks straight, but close enough so that it looks like a circular arc.





At this point on the path, a circle of radius ρ is tangent and has the same curvature as the path.

At point A let's attach our tangential and normal unit vectors $\vec{e}_t$ and $\vec{e}_n$.



$\vec{e}_n$ points towards the center of our circle.

Let's also define the angle θ such that the increment of arclength near point A is given as

$$ds = \rho\, d\theta$$

This is a total differential relationship and therefore:

$$\boxed{\frac{ds}{d\theta} = \rho \quad \text{and} \quad \frac{d\theta}{ds} = \frac{1}{\rho} \quad \text{near } A}$$

Also, we can write the (x,y) coordinates of A as

$$x = \rho\cos\theta \quad \text{and} \quad y = \rho\sin\theta$$

We are interested in $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{d^2x}{ds^2}$ and $\frac{d^2y}{ds^2}$.

Apply the chain rule: $\frac{dx}{ds} = \frac{dx}{d\theta}\frac{d\theta}{ds} = \frac{1}{\rho}\frac{dx}{d\theta}$

$$\frac{dy}{ds} = \frac{dy}{d\theta}\frac{d\theta}{ds} = \frac{1}{\rho}\frac{dy}{d\theta}$$

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\left(\frac{dx}{ds}\right) = \frac{d}{d\theta}\left(\frac{dx}{ds}\right)\frac{d\theta}{ds} = \frac{d}{d\theta}\left(\frac{1}{\rho}\frac{dx}{d\theta}\right)\frac{1}{\rho}$$

$$= \frac{1}{\rho^2}\frac{d^2x}{d\theta^2}$$

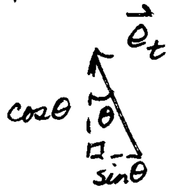
Similarly: $\frac{d^2y}{ds^2} = \frac{1}{\rho^2}\frac{d^2y}{d\theta^2}$

$$x = \rho\cos\theta \rightarrow \frac{dx}{d\theta} = -\rho\sin\theta \rightarrow \frac{d^2x}{d\theta^2} = -\rho\cos\theta$$

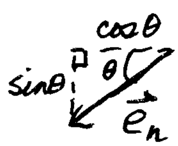
$$y = \rho\sin\theta \rightarrow \frac{dy}{d\theta} = \rho\cos\theta \rightarrow \frac{d^2y}{d\theta^2} = -\rho\sin\theta$$

$$\therefore \boxed{\begin{array}{ll} \frac{dx}{ds} = -\sin\theta & \frac{dy}{ds} = \cos\theta \\ \frac{d^2x}{ds^2} = -\frac{1}{\rho}\cos\theta & \frac{d^2y}{ds^2} = -\frac{1}{\rho}\sin\theta \end{array}}$$

From the geometry of $\vec{e}_t$ and $\vec{e}_n$



$\rightarrow \vec{e}_t = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$



$\rightarrow \vec{e}_n = -\cos\theta\,\vec{i} - \sin\theta\,\vec{j}$

$$\therefore \quad \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} = \vec{e}_t$$ As shown before

$$\frac{d^2x}{ds^2}\vec{i} + \frac{d^2y}{ds^2}\vec{j} = \frac{1}{\rho}\left(-\cos\theta\,\vec{i} - \sin\theta\,\vec{j}\right) = \frac{1}{\rho}\vec{e}_n$$

$$\therefore \quad \boxed{\vec{a} = \ddot{s}\,\vec{e}_t + \frac{\dot{s}^2}{\rho}\vec{e}_n \qquad \text{From page 92}}$$

Recap: $\vec{v} = \dot{s}\,\vec{e}_t = v\,\vec{e}_t$

$$\vec{a} = \ddot{s}\,\vec{e}_t + \frac{\dot{s}^2}{\rho}\vec{e}_n = \underbrace{a_t}_{\dot{v}}\vec{e}_t + \underbrace{\frac{v^2}{\rho}}_{a_n}\vec{e}_n$$

For path problems, including the special case of straight paths, i.e. rectilinear motion, in many cases we don't know $s(t)$ but rather $v(s)$ or equivalently $v(x)$ for linear motion. In these cases it is useful to use the following.

$$v = v(s) \rightarrow a_t = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \dot{s}\frac{dv}{ds} = v\frac{dv}{ds}$$

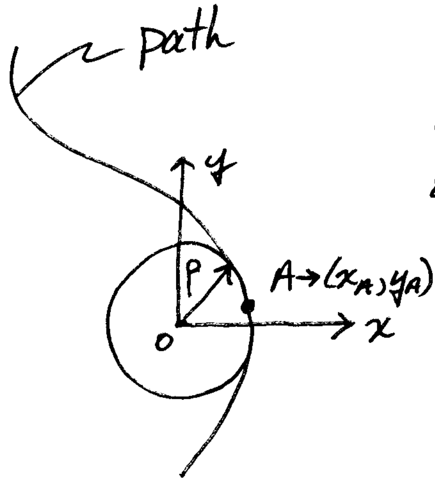
Or for linear motion $v = v(x) \rightarrow a = v\dfrac{dv}{dx}$

Then if a_t is known, e.g. a_t = constant or in general $a_t = a_t(s)$, then

$$v\frac{dv}{ds} = a_t(s) \rightarrow \int_{v_i}^{v_f} v\,dv = \int_{s_i}^{s_f} a_t(s)\,ds$$

Radius of Curvature

Derivation of the formula for radius of curvature.



Without loss of generality we can place our coordinate origin at the center of the circle we are looking for.

We are looking for the circle that passes through A, is tangent to the path at A and has the same second derivative as the path at A.

Assume we know the equation of the path $y = y(x)$.

Near A a Taylor series expansion of the path equation gives

$$y(x) = y_A + \left.\frac{dy}{dx}\right|_A (x - x_A) + \frac{1}{2}\left.\frac{d^2y}{dx^2}\right|_A (x - x_A)^2 + \ldots$$

The equation of our circle is

$$x^2 + y^2 = \rho^2$$

but A is on the circle $\rightarrow$ $x_A^2 + y_A^2 = \rho^2$

Circle equation $\rightarrow$ $y = \pm\sqrt{\rho^2 - x^2}$

$$\frac{dy}{dx} = \mp \frac{x}{\sqrt{\rho^2-x^2}} \quad , \quad \frac{d^2y}{dx^2} = \mp \frac{1}{\sqrt{\rho^2-x^2}} \pm \frac{x^2}{(\rho^2-x^2)^{3/2}}$$

$$\underbrace{\left.\frac{dy}{dx}\right|_A}_{y'} = \mp \frac{x_A}{\sqrt{\rho^2-x_A^2}} \quad , \quad \underbrace{\left.\frac{d^2y}{dx^2}\right|_A}_{y''} = \mp \frac{1}{\sqrt{\rho^2-x_A^2}} \pm \frac{x_A^2}{(\rho^2-x_A^2)^{3/2}}$$

$$\therefore\ y'' = \frac{1}{x_A} y' \pm \frac{1}{\sqrt{\rho^2-x_A^2}} y'^2$$

also $\dfrac{x_A^2}{\rho^2-x_A^2} = y'^2 \rightarrow x_A^2 = \dfrac{\rho^2 y'^2}{1+y'^2}$

$$\therefore\ y'' = \pm \frac{\sqrt{1+y'^2}}{\rho y'} y' \pm \frac{1}{\sqrt{\rho^2 - \frac{\rho^2 y'^2}{1+y'^2}}} y'^2$$

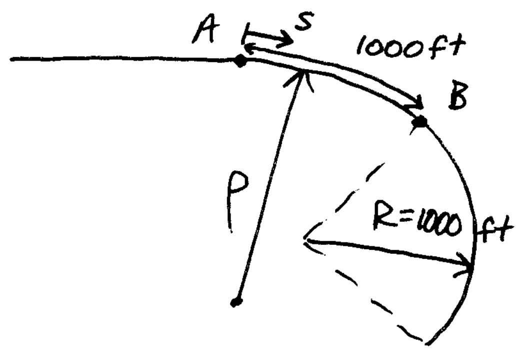
$$= \pm \frac{\sqrt{1+y'^2}}{\rho} \pm \frac{1}{\rho\sqrt{\frac{1}{1+y'^2}}} y'^2$$

$$y'' = \frac{1}{\rho}\left[\sqrt{1+y'^2}\,(\pm 1 \pm y'^2)\right]$$

By definition $\rho > 0 \rightarrow \rho = \dfrac{(1+y'^2)^{3/2}}{|y''|}$ If $y''<0$ use $-$ $y''>0$ use $+$ gives this

$$\therefore \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \quad \text{\& by analogy} \quad \rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left|\frac{d^2x}{dy^2}\right|}$$

Problem 13.21



Curvature of AB is increased linearly with distance traveled: $\frac{1}{\rho} = ks$ where $k = 1 \times 10^{-6}\ ft^{-2}$

A train enters at A with the speed of 60 ft/s and decelerates at a constant rate to a stop at B. Calculate the magnitude of the maximum acceleration and the value of s where it occurs.

$$\vec{a} = a_t \vec{e}_t + a_n \vec{e}_n = \dot{v}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n$$

$$\dot{v} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v = \text{constant}$$

$$\int_{v_A}^{v_B} v\,dv = \int_{s_A}^{s_B} \text{constant}\ ds$$

$$\frac{1}{2}v^2\Big]_{60\,ft/s}^{0} = \text{constant}\cdot s\Big]_{0}^{1000\,ft}$$

$$-1800\ \frac{ft^2}{s^2} = 1000ft\cdot\text{constant}$$

$$\text{constant} = -1.8\ ft/s^2$$

$$\therefore\ \dot{v} = -1.8\ ft/s^2$$

Now we need $v = v(s)$.

Go back to our $\dot{v}$ equation.

$$\dot{v} = v\frac{dv}{ds} = -1.8 \text{ ft/s}^2$$

$$\int_{60}^{v} v' dv' = \int_0^s -1.8 \, ds'$$

$$\frac{1}{2} v'^2 \Big]_{60}^{v} = -1.8s' \Big]_0^s$$

$$\frac{1}{2} v^2 - 1800 = -1.8s$$

$$\therefore \quad v = \sqrt{3600 - 3.6s}$$

$$\text{or} \quad v^2 = 3600 - 3.6s$$

$$\therefore \quad \frac{v^2}{\rho} = v^2 \cdot \frac{1}{\rho} = v^2 \cdot ks = 1\times10^{-6}(3600s - 3.6s^2)$$

$$\therefore \quad \vec{a} = \left(-1.8\,\vec{e}_t + 1\times10^{-6}(3600s - 3.6s^2)\,\vec{e}_n\right)\frac{ft}{s^2}$$

Since a_t = constant a will be maximized when a_n is maximized

$$\frac{da_n}{ds} = 1\times10^{-6}(3600 - 7.2s) = 0$$

$$s = \frac{3600}{7.2} = 500 \text{ ft.}$$

$$a_n(s=500) = 0.9 \text{ ft/s}^2$$

$$\therefore \quad \boxed{a_{max} = \sqrt{1.8^2 + 0.9^2} = 2.01 \text{ ft/s}^2 \text{ at } s = 500\text{ft}}$$