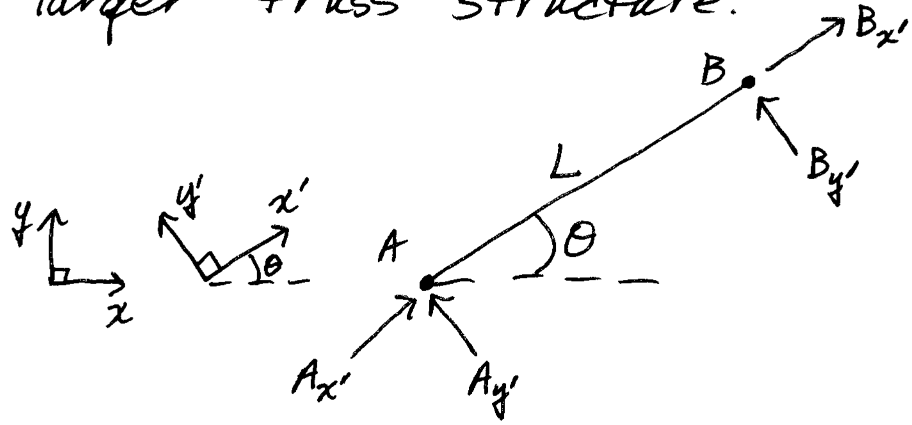
Trusses

Assumptions for the analysis of trusses.

1) Pin-ended
2) Weightless
3) No mid-span loads

Consider the truss below taken out of some larger truss structure.



The pin forces $A_{x'}$, $A_{y'}$, $B_{x'}$ and $B_{y'}$ arise due to internal reactions from the remainder of the truss structure or due to reactions from external supports.

Let's analyze the equilibrium of this truss.

$$\Sigma F_{x'} = A_{x'} + B_{x'} = 0$$
$$\Sigma F_{y'} = A_{y'} + B_{y'} = 0$$

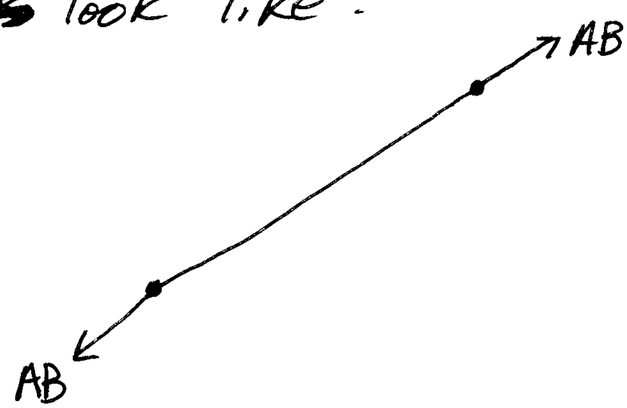
$$\Sigma M_z^A = B_{y'} L = 0 \rightarrow B_{y'} = 0 \rightarrow A_{y'} = 0 \ , \ A_{x'} = -B_{x'}$$

Hence, the forces acting on the weightless, pin-ended truss with no mid-span loads are equal and opposite and act along the line of the truss. This is true for a 3-D truss member as well.

So from now on we can use this result to simplify the FBDs we draw of trusses.

Another convention that is useful to apply is to assume that the truss is in tension. Then if our analysis determines that the truss force is negative, then the truss is actually in compression.

Then our simplified FBD of truss AB will ~~be~~ look like:

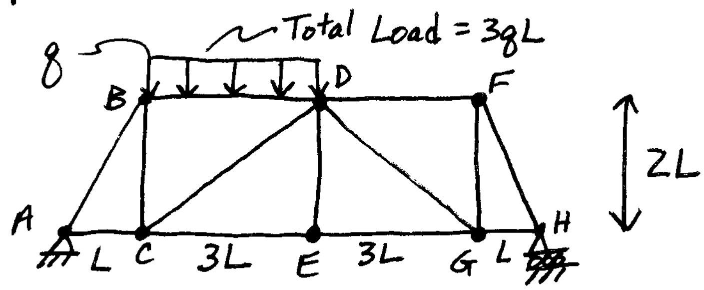


Note that pin-ended, weightless trusses with no midspan loads are <u>two-force</u> members and hence the pin forces must be equal and opposite and act through the same point.

Method of Joints

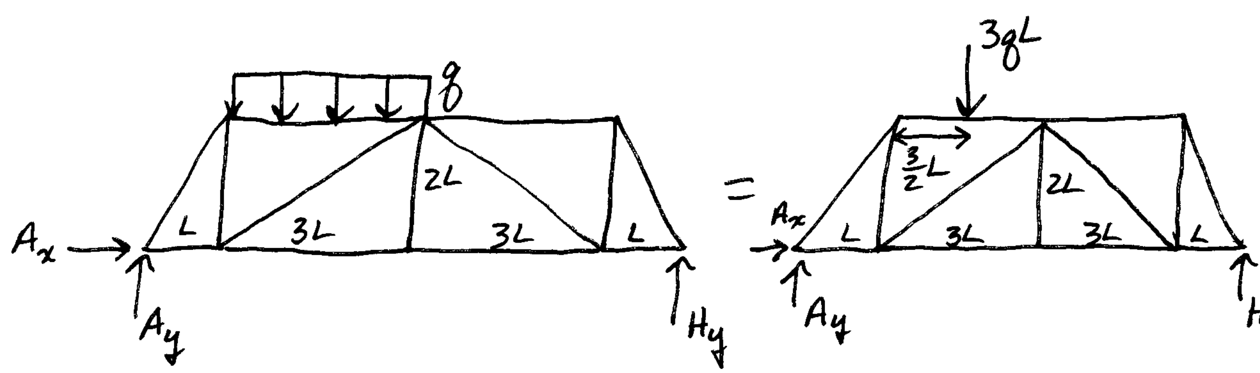
The method of joints is most useful when you are asked to determine the forces in all truss members of a system. Take the following system as an example.

Ex:



Right off the bat it seems we have a problem. Member BD has mid-span loads! This is OK because we can take BD out of the system, analyze it separately, and apply the appropriate forces to the remainder of the system.

First, let's analyze the equilibrium of the entire structure.



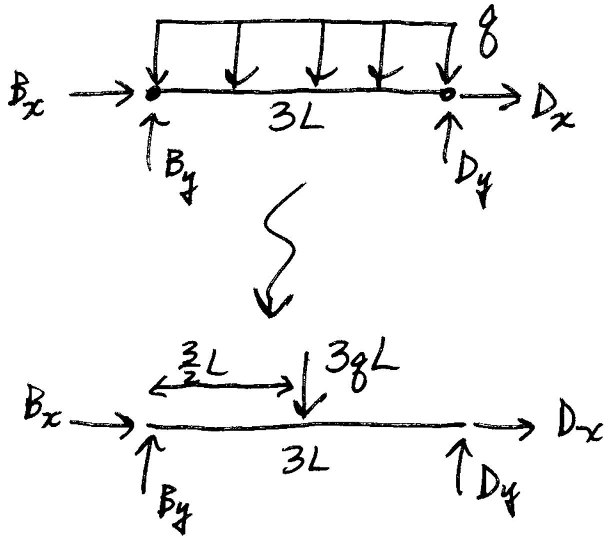
$\Sigma F_x = \boxed{A_x = 0}$

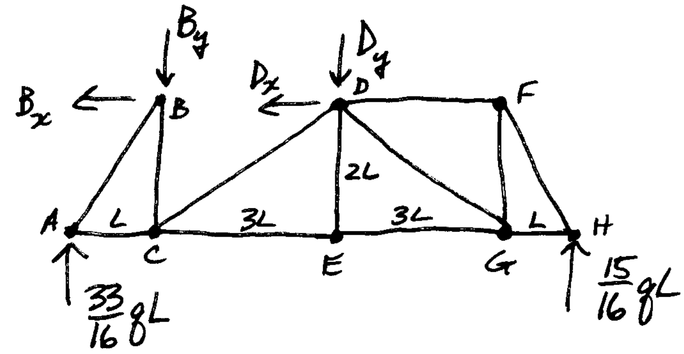
$\Sigma F_y = A_y + H_y - 3qL = 0$

$\Sigma M_z^A = 8L\,H_y - \frac{5}{2}L\,3qL = 0$

$\therefore \boxed{H_y = \frac{15}{16}qL\ ,\quad A_y = \frac{33}{16}qL}$

Now let's break the system up into BD and the rest.





$\sum F_x = B_x + D_x = 0 \rightarrow B_x = -D_x$ still unknown

$\sum F_y = B_y + D_y - 3qL = 0$

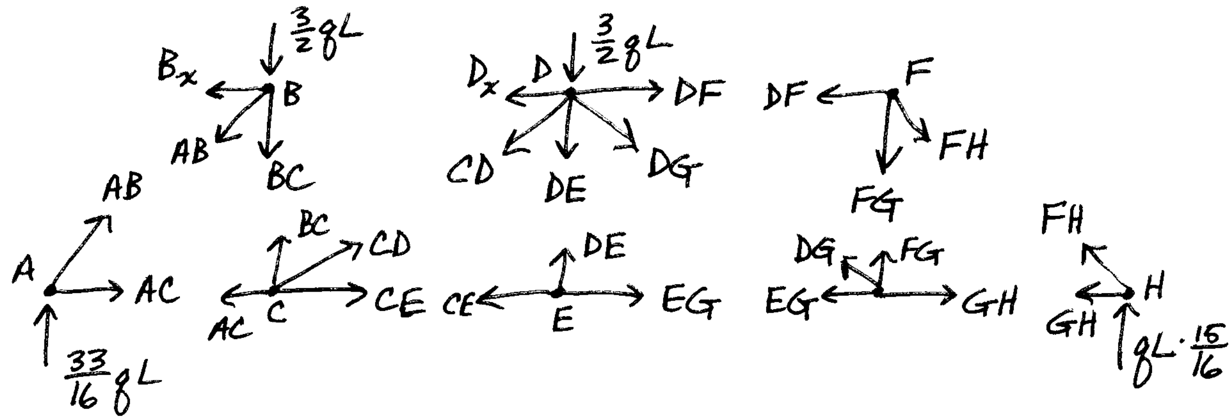
$\sum M_z^B = 3L D_y - \frac{3}{2}L\,3qL = 0 \rightarrow D_y = \frac{3}{2}qL \,,\; B_y = \frac{3}{2}qL$

Now to the truss structure.

The method of joints involves analyzing the equilibrium of each of the joints in the truss. Since we know that all weightless, pin-ended trusses/struts with **no** mid-span loads are 2-force members, **and** we will use the convention that all struts are in tension the FBD of each joint is very easy to draw. Each strut applies an outward force to the joint along the line of the strut.

Also, always remember to include any other external forces applied to the joint.

Let's go through each joint in our example.



Note from inspection of joint E, the force in member DE = 0, i.e. DE is a zero ~~force~~ force member.

Now we go through $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for each joint. $\Sigma M_z = 0$ is satisfied trivially for each joint since all forces acting on the joint have the same point of application.

A

$$\Sigma F_x = AC + AB \underbrace{\frac{L}{\sqrt{L^2+(2L)^2}}} = AC + \frac{1}{\sqrt{5}}AB = 0$$

from the geometry of triangle ABC, this is the cosine of the angle between AB and AC.

$$\Sigma F_y = \frac{33}{16}qL + \frac{2}{\sqrt{5}}AB = 0 \rightarrow \boxed{AB = -\frac{33\sqrt{5}}{32}qL \,,\, AC = \frac{33}{32}qL}$$

B

$$\Sigma F_x = -B_x - AB\frac{1}{\sqrt{5}} = -B_x + \frac{33}{32}qL = 0 \rightarrow \boxed{B_x = \frac{33}{32}qL}$$

Recall $D_x = -B_x \rightarrow \boxed{D_x = \frac{-33}{32}qL}$

$$\Sigma F_y = -\frac{3}{2}qL - \frac{2}{\sqrt{5}}AB - BC = 0$$
$$-\frac{3}{2}qL + \frac{33}{16}qL - BC = 0 \rightarrow \boxed{BC = \frac{9}{16}qL}$$

C

$$\Sigma F_x = -AC + CD\frac{3}{\sqrt{13}} + CE = 0$$
$$-\frac{33}{32}qL + \frac{3}{\sqrt{13}}CD + CE = 0 \rightarrow \boxed{CE = \frac{60}{32}qL = \frac{15}{8}qL}$$

$$\Sigma F_y = BC + CD\frac{2}{\sqrt{13}} = 0$$
$$\frac{9}{16}qL + \frac{2}{\sqrt{13}}CD = 0 \rightarrow \boxed{CD = \frac{-9\sqrt{13}}{32}qL}$$

E

$$\Sigma F_x = -CE + EG = 0 \rightarrow \boxed{EG = CE = \frac{15}{8}qL}$$

$$\Sigma F_y = \boxed{DE = 0}$$

D

$$\Sigma F_x = -D_x + DF - \frac{3}{\sqrt{13}}CD + \frac{3}{\sqrt{13}}DG = 0$$
$$\frac{33}{32}qL + DF + \frac{27}{32}qL + \frac{3}{\sqrt{13}}DG = 0 \rightarrow \boxed{DF = \frac{-15}{32}qL}$$

$$\Sigma F_y = -\frac{3}{2}qL - DE - \frac{2}{\sqrt{13}}CD - \frac{2}{\sqrt{13}}DG = 0$$
$$-\frac{3}{2}qL - 0 + \frac{18}{32}qL - \frac{2}{\sqrt{13}}DG = 0 \rightarrow \boxed{DG = \frac{-15\sqrt{13}}{32}qL}$$

F

$$\Sigma F_x = -DF + FH\frac{1}{\sqrt{5}} = 0$$
$$\frac{15}{32}qL + \frac{1}{\sqrt{5}}FH = 0 \rightarrow \boxed{FH = \frac{-15\sqrt{5}}{32}qL}$$

$$\Sigma F_y = -FG - FH\frac{2}{\sqrt{5}} = 0$$
$$-FG + \frac{30}{32}qL = 0 \rightarrow \boxed{FG = \frac{15}{16}qL}$$

H

$$\Sigma F_x = -GH - FH\frac{1}{\sqrt{5}} = 0$$
$$-GH + \frac{15}{32}qL = 0 \rightarrow \boxed{GH = \frac{15}{32}qL}$$

We have now solved for all truss forces. The remaining equilibrium equation for joint H and those for joint G can be used to check our solution.

$$\Sigma F_y = \frac{15}{16}qL + FH\frac{2}{\sqrt{5}} = 0$$
$$\frac{15}{16}qL - \frac{15\cdot 2}{32}qL = 0 \checkmark$$

G

$$\Sigma F_x = GH - EG - DG\frac{3}{\sqrt{13}} = 0$$
$$\frac{15}{32}qL - \frac{60}{32}qL + \frac{45}{32}qL = 0 \checkmark$$

$$\Sigma F_y = FG + DG\frac{2}{\sqrt{13}} = 0$$
$$\frac{15}{16}qL - \frac{15\cdot 2}{32}qL = 0 \checkmark$$

So our solution checks out.

Note that if $q > 0$ then struts with $-$ signs are in <u>compression</u> and those without are in <u>tension</u>.

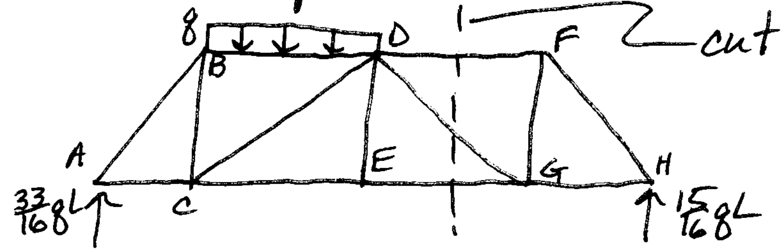
As you can see this process of solving the equilibrium equations at each joint can be rather tedious. However, if we are asked to find the forces in every strut then this method is as good as any. If we are only asked for the forces in a few members, then another method called the method of sections is preferred.

Method of Sections

In our previous problem what if we were only asked for the forces in DF, DG and EG?

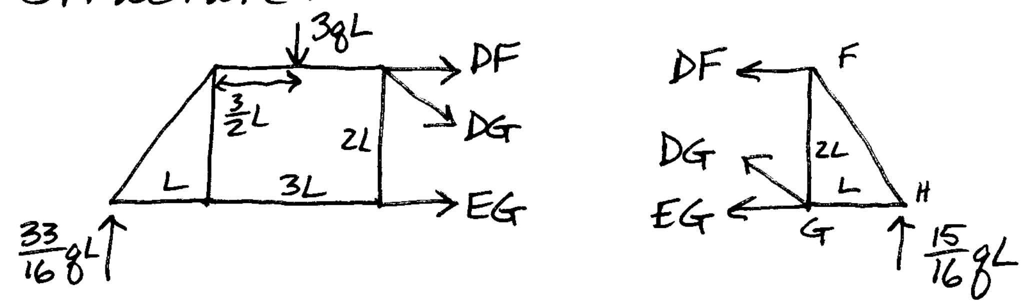
First, we would analyze the entire structure as on page 58 to find that $A_x = 0$, $A_y = \frac{33}{16} qL$ and $H_y = \frac{15}{16} qL$.

Next, we make a cut through the structure in order to "expose" the forces DF, DG and EG.



cut (does not need to be a straight line)

Now we can analyze either side of the structure.



Note that we have assumed <u>tension</u> in each strut and then the force that the strut applies to the cut is always directed outward from the cut.

Let's analyze the right FBD.

$$\Sigma F_x = -EG - DF - DG\frac{3}{\sqrt{13}} = 0$$

$$\Sigma F_y = \frac{15}{16}qL + DG\frac{2}{\sqrt{13}} = 0 \rightarrow \boxed{DG = \frac{-15\sqrt{13}}{32}qL}$$

$$\Sigma M_z^G = \frac{15}{16}qL \cdot L + DF \cdot 2L = 0$$

$$\therefore \boxed{DF = -\frac{15}{32}qL}$$

$$\therefore -EG + \frac{15}{32}qL + \frac{45}{32}qL = 0 \rightarrow \boxed{EG = \frac{60}{32}qL = \frac{15}{8}qL}$$

These answers agree with our method of joints solution. Note that the method of joints is just a special case of the method of sections with all of the "cuts" as circles around each joint.

Friction

We will consider a simple model for dry friction that is commonly known as Coulomb friction.

The Coulomb friction law states that

$$F \leq \mu_s N \quad \text{for the static case}$$

and

$$F = \mu_k N \quad \text{for the moving case}$$

F is the magnitude of the force of friction. The direction of F is always parallel to the plane of contact. For impending sliding or for the case of motion the friction force is in the opposite direction of impending motion / motion.

N is the normal force. It represents the contact force.

μ_s = coefficient of static friction

μ_k = coefficient of kinematic friction (dynamic)

Typical values for μ_s and μ_k are:

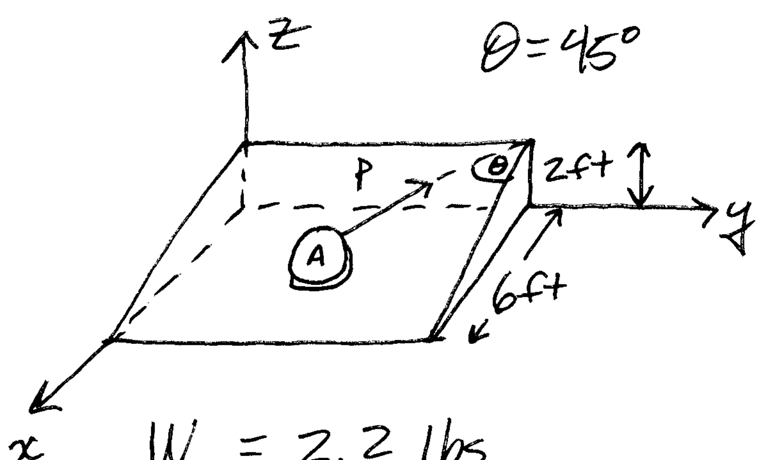
	μ_s	μ_k
Steel / Steel	0.78	0.42
Al / mild steel	0.61	0.47
Teflon / Steel	0.04	very very close to zero
Nickle / Nickle	1.10	0.53
Cu / Cast Iron	1.05	0.29
Metal / Wood	0.6	0.4
Wood / Wood	0.6	0.5
Metal / Stone	0.7	0.4
Rubber Tires / Dry Pavement	0.9	0.8

* For impending sliding the equality holds. i.e.

$$\boxed{F = \mu_s N} \quad \text{* impending sliding.}$$

Example Problem 7.23

$\theta = 45°$

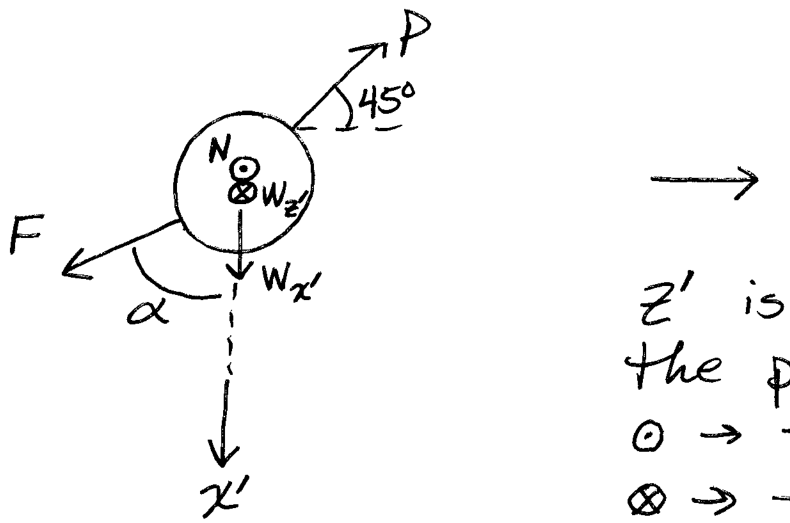


Determine the magnitude of P that causes impending sliding. Determine the direction of impending sliding.

$W_A = 2.2$ lbs.

$\mu_s = 0.4$

Let's draw a FBD of A by looking down on the plane of the incline. Let's also use a new coordinate system with y' in the same direction as y, x' down the incline, and z' perpendicular to the incline.



z' is out of the page.

$\odot \rightarrow +\ z'$ direction

$\otimes \rightarrow -\ z'$ direction

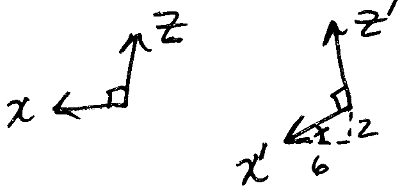
The normal force N must act perpendicular to the plane and therefore all of it acts in the z' direction.

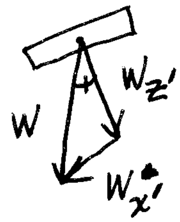
F must act in the plane, but we don't know at what angle.

Since we are analyzing sliding we do know the magnitude of F equals $\mu_s N$.

The weight W will have components $W_{x'}$ and $W_{z'}$ in the x' and z' directions.

Let's analyze W in the $x'-z'$ plane





$$W_{z'} = W \frac{6}{\sqrt{40}}$$ ← cosine of the angle

$$W_{x'} = W \frac{2}{\sqrt{40}}$$ ← sine of the angle

Let's now analyze the equilibrium of the puck in x', y, z' coordinates $(y = y')$

$$\Sigma F_{x'} = -P \frac{\sqrt{2}}{2} + W_{x'} + F \cos\alpha = 0$$

$$\Sigma F_{y} = P \frac{\sqrt{2}}{2} - F \sin\alpha = 0$$

$$\Sigma F_{z'} = N - W_{z'} = 0$$

$$\therefore \quad N = W \frac{6}{\sqrt{40}}$$

$$P = \sqrt{2}\, F \sin\alpha \quad \text{or} \quad F = \frac{P}{\sqrt{2} \sin\alpha}$$

$$-P \frac{\sqrt{2}}{2} + W \frac{2}{\sqrt{40}} + \frac{P}{\sqrt{2}} \frac{\cos\alpha}{\sin\alpha} = 0$$

$$\therefore \quad P\left(\frac{\cos\alpha}{\sin\alpha}-1\right)\frac{1}{\sqrt{2}} = -W\frac{2}{\sqrt{40}}$$

$$\therefore \quad P = -W\frac{2\sqrt{2}}{\sqrt{40}\left(\frac{\cos\alpha}{\sin\alpha}-1\right)} = W\frac{2\sqrt{2}}{\sqrt{40}\left(1-\frac{\cos\alpha}{\sin\alpha}\right)}$$

But we have impending sliding, so

$$N = F/\mu_s = \frac{P}{\sqrt{2}\,\mu_s \sin\alpha} = W\frac{6}{\sqrt{40}}$$

$$P = W\frac{6\sqrt{2}\,\mu_s \sin\alpha}{\sqrt{40}}$$

$$\text{so} \quad \cancel{W}\frac{6\cancel{\sqrt{2}}\,\mu_s \sin\alpha}{\cancel{\sqrt{40}}} = \cancel{W}\frac{2\cancel{\sqrt{2}}}{\cancel{\sqrt{40}}\left(1-\frac{\cos\alpha}{\sin\alpha}\right)}$$

$$\therefore \quad 3\mu_s = \frac{1}{\sin\alpha - \cos\alpha}$$

$$\text{or} \quad \sin\alpha - \cos\alpha = \frac{1}{3\mu_s} = \frac{1}{1.2}$$

Solve numerically → $\boxed{\alpha = 81.1^\circ}$

$$\text{Then} \quad P = W\frac{6\sqrt{2}\,(0.4)\sin 81.1^\circ}{\sqrt{40}} = \boxed{0.53W = P}$$

$$= 0.53 \cdot 2.2 = 1.17 \text{ lbs.}$$

$$\text{check with} \quad P = W\frac{2\sqrt{2}}{\sqrt{40}\left(1-\frac{\cos\alpha}{\sin\alpha}\right)} = 0.53W \ \checkmark$$

Note if you found the other root for α, i.e. $\alpha = -171.1^\circ$ then you would have

$$P = W\frac{6\sqrt{2}\,(0.4)\sin(-171.1^\circ)}{\sqrt{40}} = -0.083W$$

, i.e. P acts in the opposite direction.

We actually could have solved the equation for α on the last page more directly. Recall that we had the equation:

$$\sin\alpha - \cos\alpha = \frac{1}{1.2}$$

Square both sides

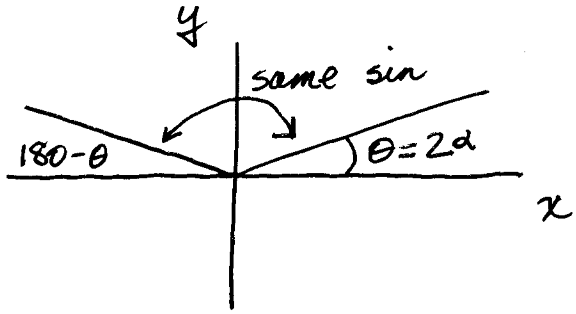
$$\rightarrow \sin^2\alpha + \cos^2\alpha - 2\sin\alpha\cos\alpha = \frac{1}{1.44}$$

$$1 - \sin 2\alpha = \frac{1}{1.44}$$

$$\therefore \sin 2\alpha = 1 - \frac{1}{1.44}$$

$$\alpha = \frac{1}{2}\arcsin\left(1 - \frac{1}{1.44}\right) = \frac{1}{2}(17.8^\circ) = 8.9^\circ$$

$$\text{or } \frac{1}{2}(180^\circ - 17.8^\circ) = 81.1^\circ$$



When using the arcsin we have to be careful to consider which solution solves the physical problem in question. We also have to be careful when we square an equation.

Note $(\sin\alpha - \cos\alpha)^2 = \left(\frac{1}{1.2}\right)^2 = \left(\frac{-1}{1.2}\right)^2$

Check $\alpha = 8.9^\circ$ $\quad \sin 8.9^\circ - \cos 8.9^\circ = -0.833 = \frac{-1}{1.2}$ X

$\boxed{\alpha = 81.1^\circ}$ $\quad \sin 81.1^\circ - \cos 81.1^\circ = 0.833 = \frac{1}{1.2}$ ✓