Equilibrium Analysis and Free Body Diagrams

There are two key components to every equilibrium analysis.

1) Equilibrium equations

Vector form: $\sum \vec{F} = 0$

$\sum \vec{M}^A = 0$

Scalar form:

$\sum F_x = 0$ * $\sum M_x^A = 0$

$\sum F_y = 0$ * $\sum M_y^A = 0$

$\sum F_z = 0$ $\sum M_z^A = 0$ *

* Necessary for 2-D (planar) analyses.

2) Correct and well-labeled Free Body Diagrams.
A correct free body diagram has all possible reaction forces drawn at locations were external supports or connections to other components have been removed.

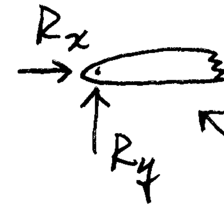
If a previous analysis has indicated that a connection force or support force is actually zero, then and only then is it acceptable to omit that force from a subsequently drawn FBD.

Types of supports (2-D)

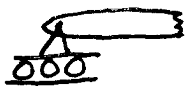
1) 

frictionless pin

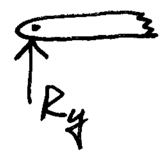
Simple support →



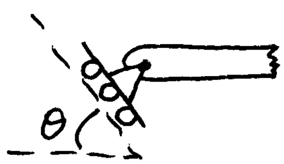
No couple

2) 

Simple support on rollers →

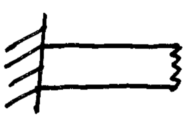


Free to move in x-dir

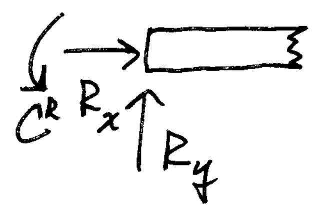
3) 

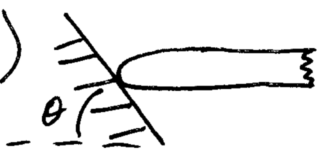
No reaction force parallel to rollers →

right angle

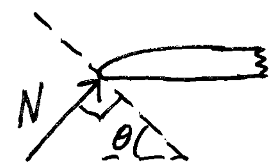
4) 

Built-in or cantelever support →



5) 

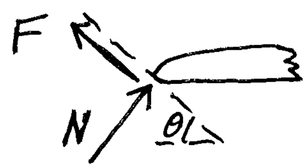
Frictionless surface →



Just like rollers. Only a normal force N and no tangential friction force.

6) 

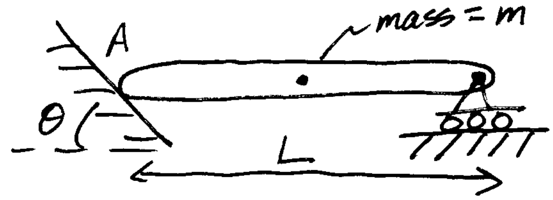
Surface with friction →



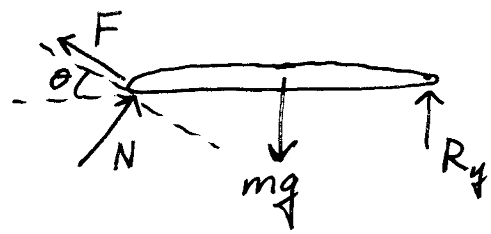
Normal force N & tangential friction force F. Plus $F \le \mu_s N$.

6 con'd) With friction we have an extra step. After the analysis we must check that $F \le \mu_s N$. Where μ_s is the static coefficient of friction.

Ex)



FBD:



$$\Sigma F_x = N\sin\theta - F\cos\theta = 0 \rightarrow F = N\frac{\sin\theta}{\cos\theta}$$

$$\Sigma F_y = N\cos\theta + F\sin\theta - mg + R_y = 0$$

$$\Sigma M_z^A = -mg\frac{L}{2} + R_y L = 0$$

$$\therefore R_y = \frac{mg}{2}$$

$$\longrightarrow N\cos\theta + N\frac{\sin^2\theta}{\cos\theta} - mg + \frac{mg}{2} = 0$$

$$N\cos^2\theta + N\sin^2\theta = \frac{mg}{2}\cos\theta$$

$$\therefore N = \frac{mg}{2}\cos\theta \quad \text{and} \quad F = \frac{mg}{2}\sin\theta$$

Check if $F \le \mu_s N$

$$\frac{mg}{2}\sin\theta \le \mu_s \frac{mg}{2}\cos\theta \quad \text{true if } \mu_s \ge \frac{\sin\theta}{\cos\theta}$$

$\therefore$ If $\mu_s \ge \tan\theta$ the beam will not slide.

7) 

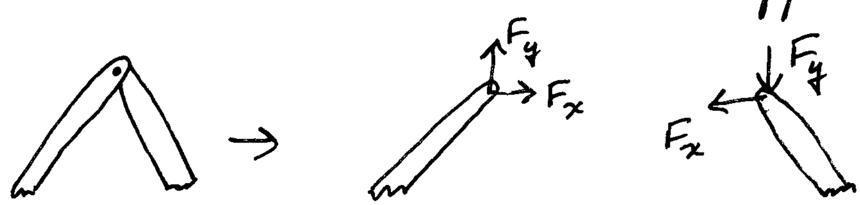
Cable → 

* Pick correct direction to draw. Assume tension.

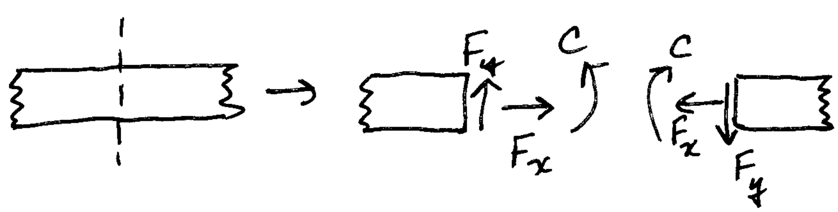
After the analysis you must check to see that $T > 0$. Cables cannot support compressive loads.

8) Internal connections → always draw equal and opposite

Pin :



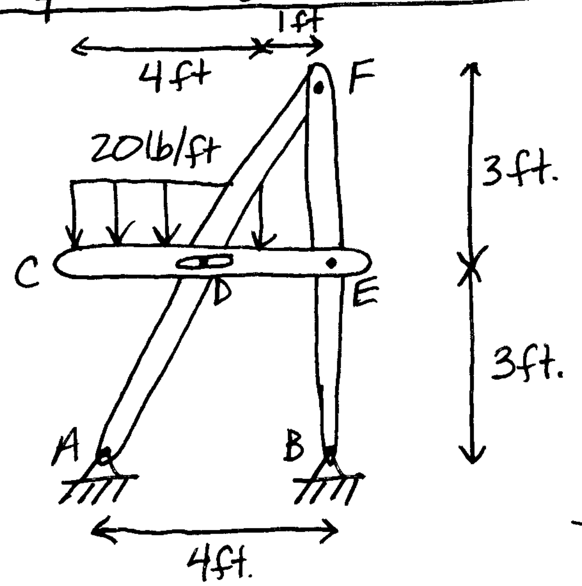
Internal beam cut :



In general, each of these types of supports has an analog in 3-D. In this course we will focus on 2-D/planar equilibrium analysis.

Example Problem 4.10

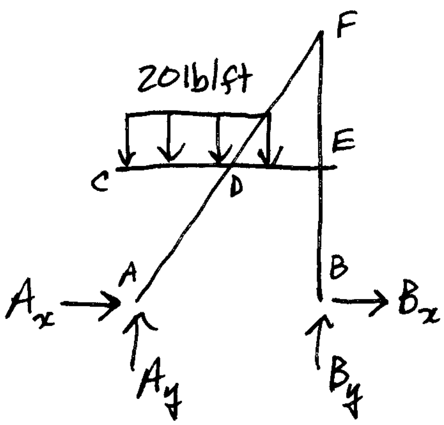
Assume weightless components.

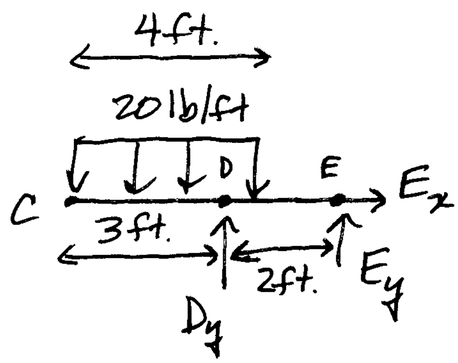


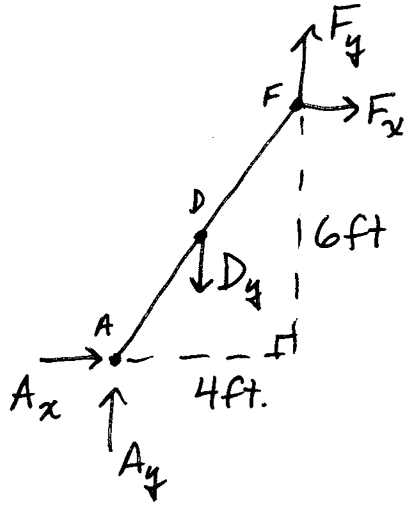
Note AFB and DFE are similar triangles, therefore DE is 2ft long.

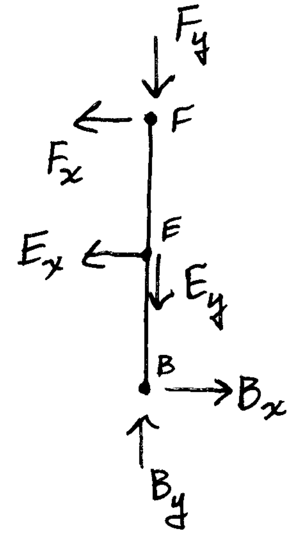
Note there is a horizontal frictionless slot at D.

Draw FBDs of the entire structure and each component.





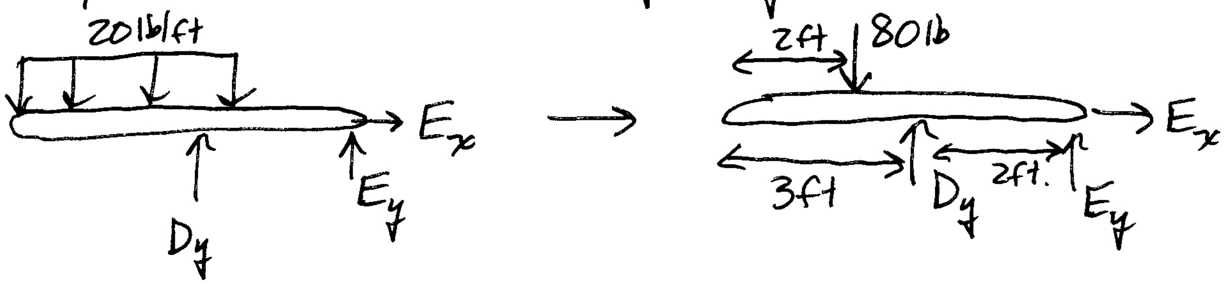




Notice that if we re-combine (i.e. add) the individual component FBDs the internal connection forces cancel out and we recover the FBD of the entire structure.

When you attempt to analyze the equilibrium of a composite structure look for the FBD with the fewest number of unknowns. For 2-D problems, if there are 3 or fewer unknowns for a given FBD then we should be able to solve for them.

In our current problem component CDE has only 3 unknowns, D_y, E_y and E_x.



$$\Sigma F_x = \boxed{E_x = 0}$$

$$\Sigma F_y = D_y + E_y - 80 = 0$$

$$\Sigma M_z^E = -2D_y + 3 \cdot 80 = 0$$

$$\therefore \boxed{D_y = 120 \text{ lbs} \;,\; E_y = -40 \text{ lbs}}$$

Now look for another remaining FBD with 3 or less remaining unknowns. In our case each of the remaining FBDs has 4 unknowns. So we will have to analyze at least 2 of them.

Bar AF

$$\Sigma F_x = A_x + F_x = 0$$
$$\Sigma F_y = A_y - 120 + F_y = 0$$
$$\Sigma M_z^A = -6F_x + 4F_y - 2\cdot 120 = 0$$

Bar BF

$$\Sigma F_x = B_x - F_x = 0 \qquad \text{note } E_x = 0$$
$$\Sigma F_y = B_y + 40 - F_y = 0 \qquad \text{note } E_y = {}^{-}40$$
$$\Sigma M_z^B = 6F_x = 0$$

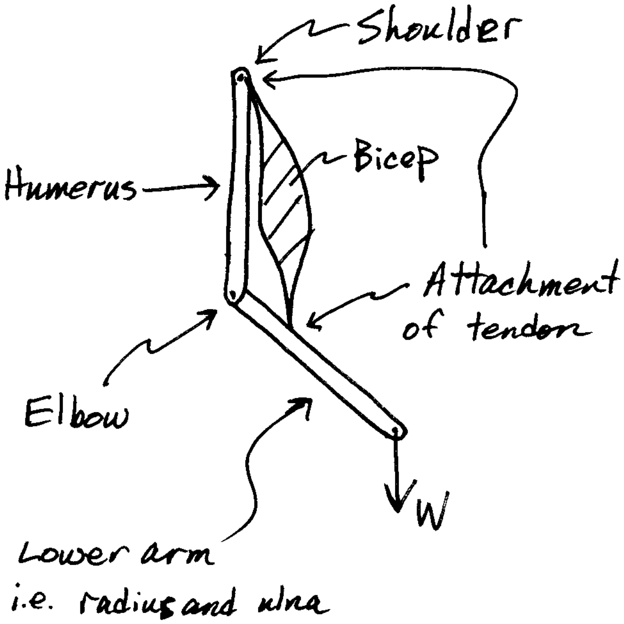
$$\therefore \boxed{F_x = 0,\ B_x = 0,\ A_x = 0,\ F_y = 60,\ A_y = 60,\ B_y = 20}$$

We can check our answer with the FBD of the entire structure.

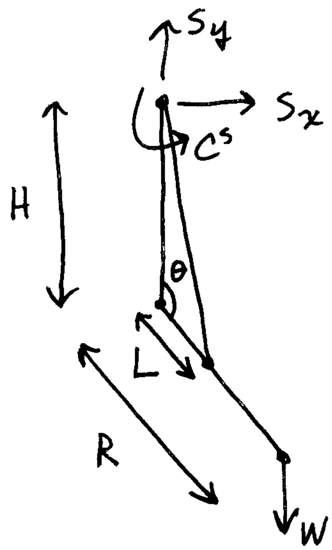
$$\Sigma F_x = A_x + B_x = 0 + 0 = 0 \checkmark$$
$$\Sigma F_y = A_y + B_y - 80 = 60 + 20 - 80 = 0 \checkmark$$
$$\Sigma M_z^B = 3\cdot 80 + {}^{-}4\cdot A_y = 240 - 240 = 0 \checkmark$$

Everything checks out.

Example: The Bicep Curl



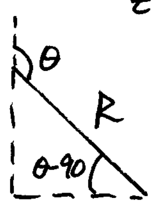
Assume we keep our humerus perpendicular to the ground. We will also assume that the bicep applies a cable-like tension between its attachment points.



$$\sum F_x = \boxed{S_x = 0}$$

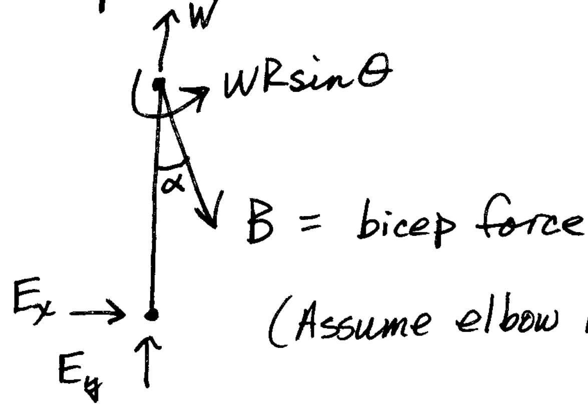
$$\sum F_y = S_y - W = 0 \rightarrow \boxed{S_y = W}$$

$$\sum M_z^S = C_s - WR\underbrace{\cos(\theta - 90^\circ)}_{\sin\theta} = 0$$



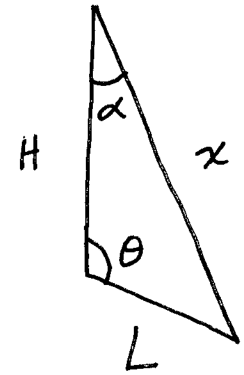
$$\therefore \boxed{C_s = WR\sin\theta}$$

But we want to determine the force in the bicep. FBD of humerus.



(Assume elbow is a frictionless hinge)

Some geometry:



Law of cosines: $x^2 = H^2 + L^2 - 2HL\cos\theta$

Law of sines: $\dfrac{x}{\sin\theta} = \dfrac{L}{\sin\alpha}$

With these two equations we can solve for α given H, L and θ.

Equilibrium of the humerus. We are really only interested in B. We can find B directly by taking moments about the elbow.

$$\Sigma M_z^E = WR\sin\theta - HB\sin\alpha = 0$$

$$\therefore \boxed{B = W\frac{R}{H}\frac{\sin\theta}{\sin\alpha}}$$

From laws of sines and cosines $\quad \dfrac{\sin\theta}{\sin\alpha} = \dfrac{x}{L} = \dfrac{\sqrt{H^2+L^2-2HL\cos\theta}}{L}$

$$\therefore \quad B = W\frac{R}{H}\frac{\sqrt{H^2+L^2-2HL\cos\theta}}{L}$$

Let's find B_{max}.

θ_{crit} occurs where $\frac{dB}{d\theta} = 0$

$$\frac{dB}{d\theta} = \frac{WR}{HL} \frac{1}{2}\left(H^2+L^2-2HL\cos\theta\right)^{-1/2} HL\sin\theta$$

$$= \frac{WR}{2} \frac{\sin\theta}{\sqrt{H^2+L^2-2HL\cos\theta}} = 0$$

This is a non-linear equation for θ. But note that it will be satisfied if $\sin\theta = 0$. This will occur if $\theta = 0°$ or $180°$, but we are looking for a solution between these 2 limits because our elbow joint is never actually at $0°$ or $180°$. Since no extremal point occurs in between, our max B must occur at either θ_{min} or θ_{max}. Check which one.

$$B(\theta=\theta_{min}) = W\frac{R}{H}\frac{\sqrt{H^2+L^2-2HL\cos\theta_{min}}}{L}$$

$$B(\theta=\theta_{max}) = W\frac{R}{H}\frac{\sqrt{H^2+L^2-2HL\cos\theta_{max}}}{L}$$

Notice that $\sqrt{H^2+L^2-2HL\cos\theta}$ is largest when $\cos\theta<0$ and $\cos\theta<0$ when $\theta>90°$. Therefore our bicep has to impart the most force at the very beginning of the curl, and lifting the weight gets easier as we go up. Usually you will see people change the angle of their humerus or use some inertia to get started on this motion.

Let's see if we can estimate the strength of the bicep muscle. To make things easy take $\theta = 90°$.

$$B = \frac{WR}{H} \frac{\sqrt{H^2+L^2}}{L} = W\frac{R}{H}\sqrt{\left(\frac{H}{L}\right)^2+1}$$

For my body $R \approx 12$ inches
$H \approx 10$ inches

I did not feel like cutting my arm open, so I don't know what L is. Let's take L to be about 1 inch. Let's also say I can support 50 lbs. when my elbow is at 90°.

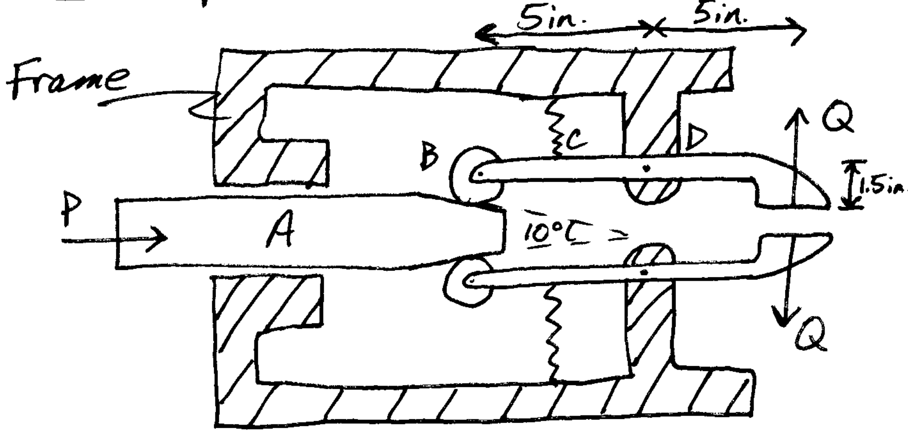
then $B = 50\,\frac{12}{10}\sqrt{10^2+1} \approx 600$ lbs

Wow! But what if I'm way off on L and it is actually 2 inches. Then,

$$B = 50\,\frac{12}{10}\sqrt{5^2+1} \approx 300 \text{ lbs}$$

Still a big number, and I actually think L is closer to 1 inch. Notice that given the same B (bicep strength) people with longer L can lift more weight. What is strange is that arms with longer L can lift more weight but don't actually look as big and muscular as those with short L.

Example Problem 4.91

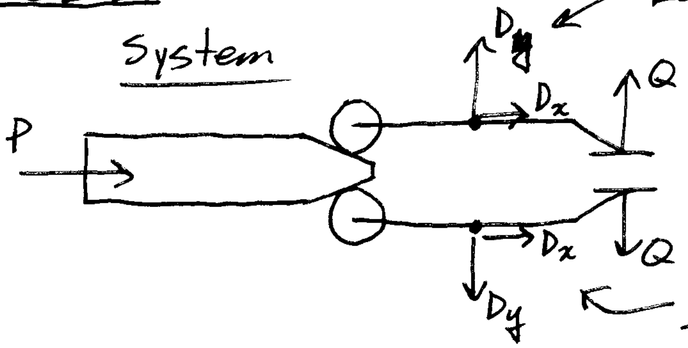


Neglect spring forces.
Determine the relationship between P and Q for the position shown.
Neglect gravity.

FBDs

Reactions due to frame.

System

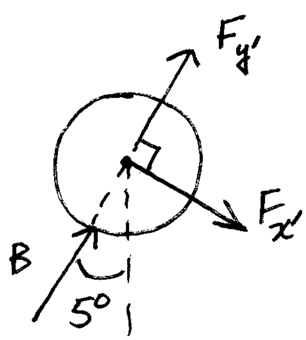


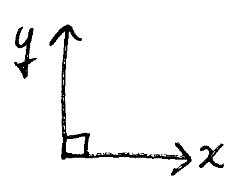
I have made these equal to the forces above due to the symmetry of the system.

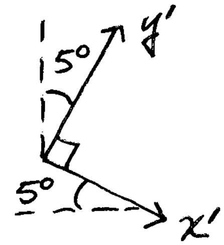
Plunger

Again, symmetry invoked. These forces are normal to the surface.

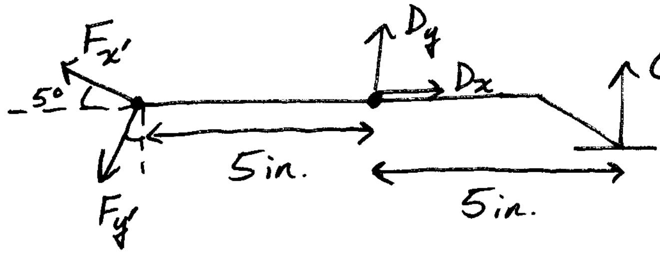
Top Wheel







Top Bar



Equilibrium Analysis (In addition to Q, let's also determine the frame reactions.)

Plunger: $\Sigma F_x = P - B\sin 5° - B\sin 5° = 0$

$$\therefore \quad B = \frac{P}{2\sin 5°}$$

Top wheel: $\Sigma F_{x'} = \boxed{F_{x'} = 0}$

$$\Sigma F_{y'} = B + F_{y'} = 0$$

$$\therefore \boxed{F_{y'} = -B = \frac{-P}{2\sin 5°}}$$

Note: This wheel is a 2-force member. There are only 2 forces acting on it, B and the pin force. For any 2-force member the forces must be equal and opposite and act through the same point.

Top Bar

$$\Sigma F_x = -\cancel{F_{x'}}^{\,0}\cos 5° - F_{y'}\sin 5° + D_x = 0$$

$$\therefore \quad D_x = F_{y'}\sin 5° = \frac{-P}{2\sin 5°}\sin 5° = \frac{-P}{2}$$

$$\Sigma F_y = \cancel{F_{x'}}^{\,0} \sin 5° - F_{y'} \cos 5° + D_y + Q = 0$$

$$\Sigma M_z^D = -5\cancel{F_{x'}}^{\,0} \sin 5° + 5F_{y'} \cos 5° + 5Q = 0$$

$$\therefore Q = -F_{y'} \cos 5° = \frac{P}{2}\frac{\cos 5°}{\sin 5°} = \frac{P}{2}\cot 5°$$

then $D_y = F_{y'} \cos 5° - Q = -2Q = -P \cot 5°$

$$\therefore \boxed{\begin{aligned} Q &= \frac{P}{2}\cot 5° = 5.7\,P \\ D_x &= \frac{-P}{2} = -0.5\,P \\ D_y &= -P\cot 5° = -11.4\,P \end{aligned}}$$