Distributed Loads

Recall two force-couple systems are equivalent if

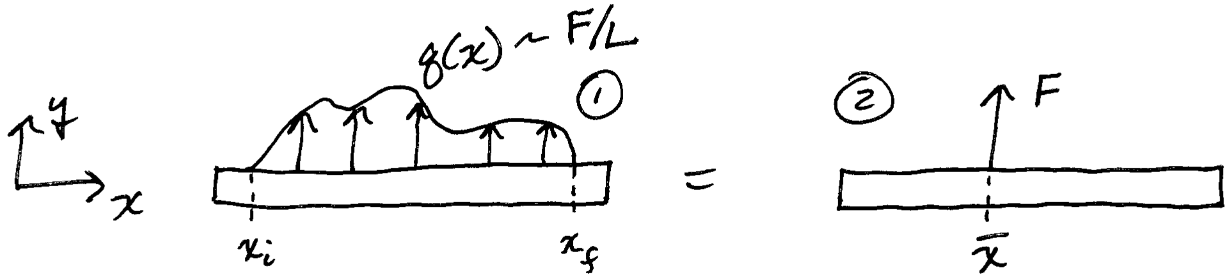
$$\Sigma \vec{F}^{①} = \Sigma \vec{F}^{②}$$

and

$$\Sigma \vec{M}_A^{①} = \Sigma \vec{M}_A^{②}$$

So far we have only dealt with point forces and couples. Distributed loads are another type of loading that we must consider as well. We will analyze loads distributed in one dimension, but the concepts can be extended to other higher-dimensional distributions as well. For example, pressure distributions are usually over 2 dimensions and gravitational forces cause 3-D load distributions.

We will focus on 1-D distributions.

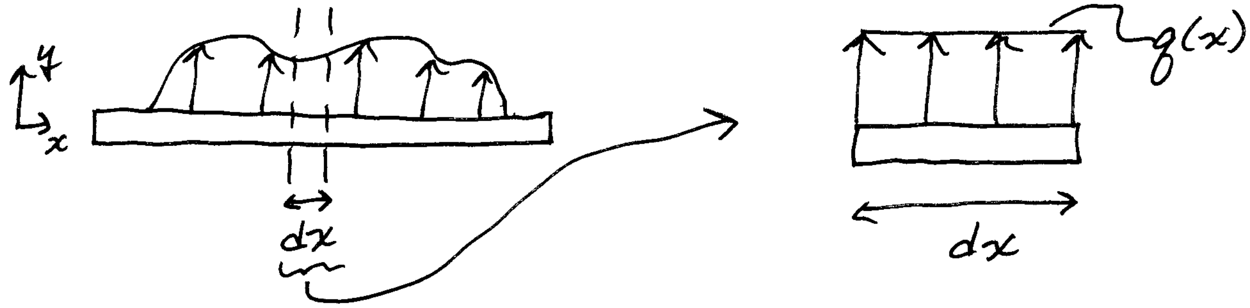


What is F? Where is $\bar{x}$?

To answer these questions we must use the relationships for equivalent systems.

First $\quad \Sigma \vec{F}^{①} = \Sigma \vec{F}^{②}$

Let's analyze system 1. First look at a small, i.e. differential, element of our beam.



If our differential element is small enough then the distributed load acting on it is constant (to first order). Then the small amount of force, dF, acting on this part of the beam is

$$dF = q(x)\,dx$$

Note 2 things here; a) this is an equation in the $\vec{j}$ direction and b) dF has dimensions of F, $q(x) \rightarrow F/L$ and $dx \rightarrow L$ $\therefore$ the dimensions on each side of the equation agree.

Now we must sum up all of the contributions to the total force along the beam, i.e.

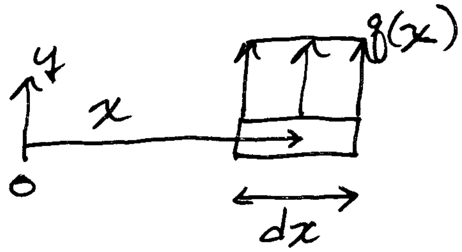
$$F = \int_{x_i}^{x_f} dF = \int_{x_i}^{x_f} q(x)\,dx$$

$$\therefore \quad \Sigma \vec{F}^{①} = \int_{x_i}^{x_f} q(x)\,dx \ \vec{j}$$

$$\Sigma \vec{F}^{②} = F \vec{j}$$

$$\therefore \quad \boxed{F = \int_{x_i}^{x_f} q(x)\,dx}$$

To determine the point of application of this force, $\bar{x}$, we must use the moment equation. We could pick an arbitrary point x_A to take moments about, but the result is exactly the same if we take moments about the origin. Let's analyze the moment about $x=0$ due to $q(x)$.



$$d\vec{M} = x\,dF\ \vec{k}$$
$$= x\, q(x)\,dx\ \vec{k}$$

$$\Sigma \vec{M}_0^{①} = \int d\vec{M} = \int_{x_i}^{x_f} x\, q(x)\,dx\ \vec{k}$$

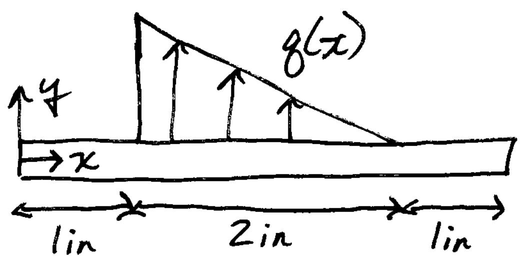
$$\therefore \vec{k} \int_{x_i}^{x_f} x\, q(x)\,dx = F \bar{x}\, \vec{k}$$

$$\therefore \quad \boxed{\bar{x} = \frac{\int_{x_i}^{x_f} x\, q(x)\,dx}{\int_{x_i}^{x_f} q(x)\,dx}}$$

Hence, the equivalent system to the force distribution $q(x)$ is the force $F = \int_{x_i}^{x_f} q(x)\,dx$ located at the point

$$\bar{x} = \frac{\int_{x_i}^{x_f} x\,q(x)\,dx}{\int_{x_i}^{x_f} q(x)\,dx}.$$

Example:



The total force due to $q(x) = 10$ lbs. Determine the distribution $q(x)$ and its equivalent point force system.

$q(x)$ is a linear distribution with $q(x) = q_{max}$ at $x = 1$ and $q(x) = 0$ at $x = 3$. There are many ways to write this distribution, e.g.

$$q(x) = mx + b$$

$$\left.\begin{aligned} q(1) &= (1\text{in})m + b = q_{max} \\ q(3) &= (\text{in})3m + b = 0 \end{aligned}\right\} \rightarrow \begin{aligned} m &= \frac{-q_{max}}{2}\frac{1}{\text{in.}} \\ b &= \frac{3}{2}q_{max}\frac{1}{\text{in}} \end{aligned}$$

$$\therefore\ q(x) = \frac{-q_{max}}{2}x + \frac{3}{2}q_{max} = \frac{-q_{max}}{2}(x - 3_{\text{in}})\frac{1}{\text{in.}}$$

$$F = \int_{x=1}^{x=3} \frac{-q_{max}}{2\,\text{in}}(x - 3_{\text{in}})\,dx = 10 \text{ lbs.}$$

$$= \frac{-q_{max}}{4\text{in}}(x - 3_{\text{in}})^2\Big]_{1\text{in}}^{3\text{in}} = 10 \text{ lbs.}$$

$$F = \frac{-q_{max}}{4\,in}\left[0 - 4\,in^2\right] = 10\,lbs \rightarrow q_{max} = 10\,{}^{lbs}/_{in}$$

$$\therefore \boxed{q(x) = -5\,{}^{lbs}/_{in^2}\,(x - 3\,in)}$$

$$M_o = \int_1^3 x \cdot -5\,{}^{lb}/_{in^2}\,(x - 3\,in)\,dx$$

$$= \int_1^3 -5\,{}^{lb}/_{in^2}\,x^2 + 15\,{}^{lb}/_{in}\,x\,dx$$

$$= -\frac{5}{3}\frac{lb}{in^2}x^3 + \frac{15}{2}\frac{lb}{in}x^2\Big]_{1\,in}^{3\,in}$$

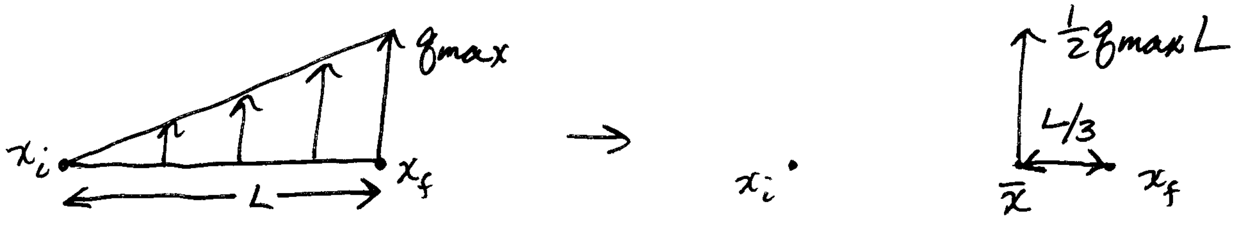
$$= -\frac{5}{3}\frac{lb}{in^2}(27\,in^3) + \frac{15}{2}\frac{lb}{in}(9\,in^2) + \frac{5}{3}\frac{lb}{in^2}(1\,in^3) + \frac{-15}{2}\frac{lb}{in}(1\,in^2)$$

$$= -45\,lb\cdot in + \frac{135}{2}\,lb\cdot in + \frac{5}{3}\,lb\cdot in + \frac{-15}{2}\,lb\cdot in$$

$$M_o = 15\tfrac{5}{3}\,lb\cdot in = 16\tfrac{2}{3}\,lb\cdot in$$

$$\therefore \boxed{\bar{x} = \frac{M_o}{F} = \frac{16\,{}^2/_3\,lb\cdot in}{10\,lb} = 1\tfrac{2}{3}\,in.}$$

In general, the Force = area under $q(x)$. The position $\bar{x}$ for a <u>triangular</u> distribution is $\frac{1}{3}$ of the distance $x_f - x_i$ from where q_{max} acts.



Internal Forces & Distributed Loads

We have now covered the analysis of forces, moments, couples and distributed loads. We will use these tools to study the equilibrium of rigid bodies.

For our analysis of internal forces we will introduce one of the most important concepts in equilibrium analysis, the <u>free body diagram</u>.

Most of you have already been initiated to the conditions for static equilibrium of a rigid body, i.e.

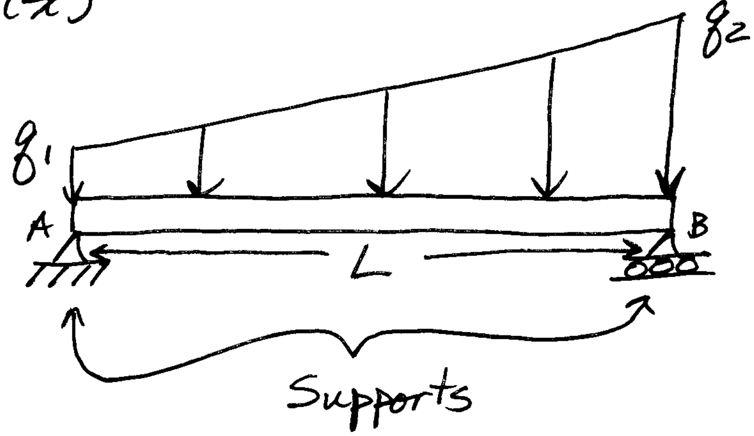
$$\Sigma \vec{F} = 0 \quad \rightarrow \quad \begin{array}{ll} \Sigma F_x = 0 & * \\ \Sigma F_y = 0 & * \\ \Sigma F_z = 0 & \end{array}$$

$$\Sigma \vec{M}_A = 0 \quad \rightarrow \quad \begin{array}{ll} \Sigma M_x^A = 0 & \\ \Sigma M_y^A = 0 & \\ \Sigma M_z^A = 0 & * \end{array}$$

* Important components for 2-D analysis.

The determination of internal forces is especially important for failure analysis.

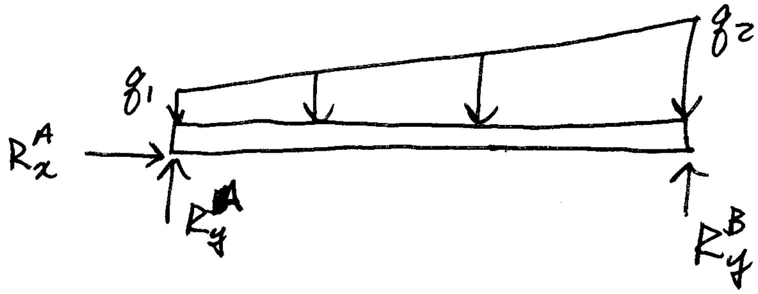
Consider the beam loaded by the distributed load $q(x)$



We would like to determine the maximum internal moment that this beam must support.

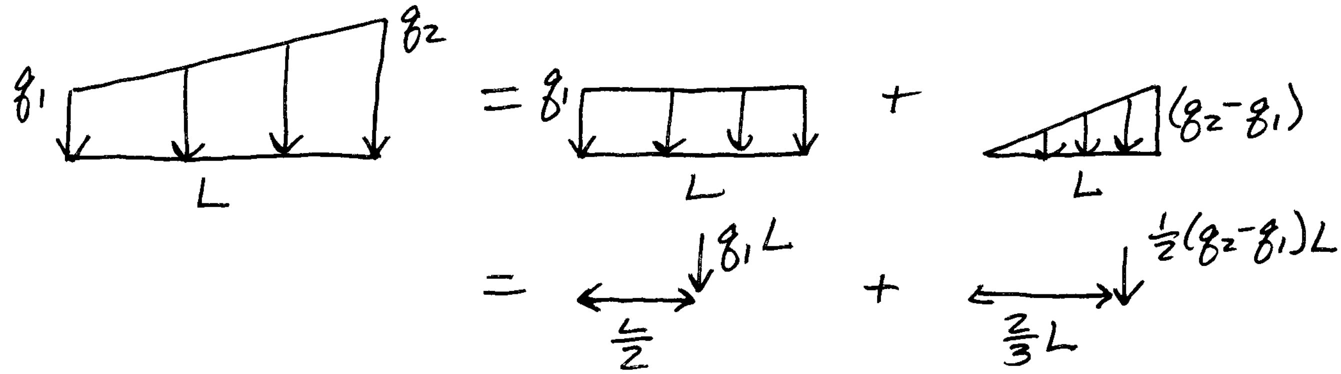
First we must analyze the conditions ~~for~~ static equilibrium of the entire structure.

Draw a free body diagram of the entire structure. A FBD replaces supports with <u>all possible</u> forces and couples that the support can apply to the structure.

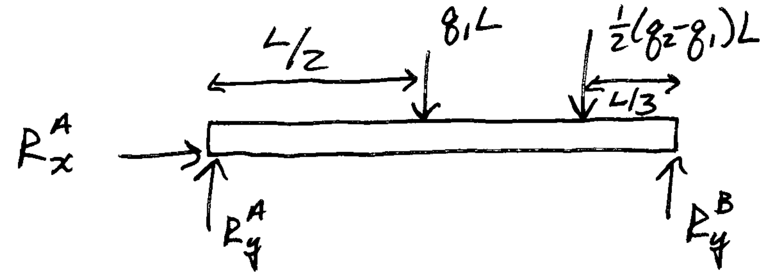


→ This support can move freely parallel to the rollers

Now analyze the distributed load and replace it with an equivalent load system.



Now we can re-draw our FBD



Now analyze the equilibrium equations.

$$\Sigma F_x = \boxed{R_x^A = 0}$$

$$\Sigma F_y = R_y^A + R_y^B - q_1 L - \frac{1}{2}(q_2 - q_1)L = 0$$

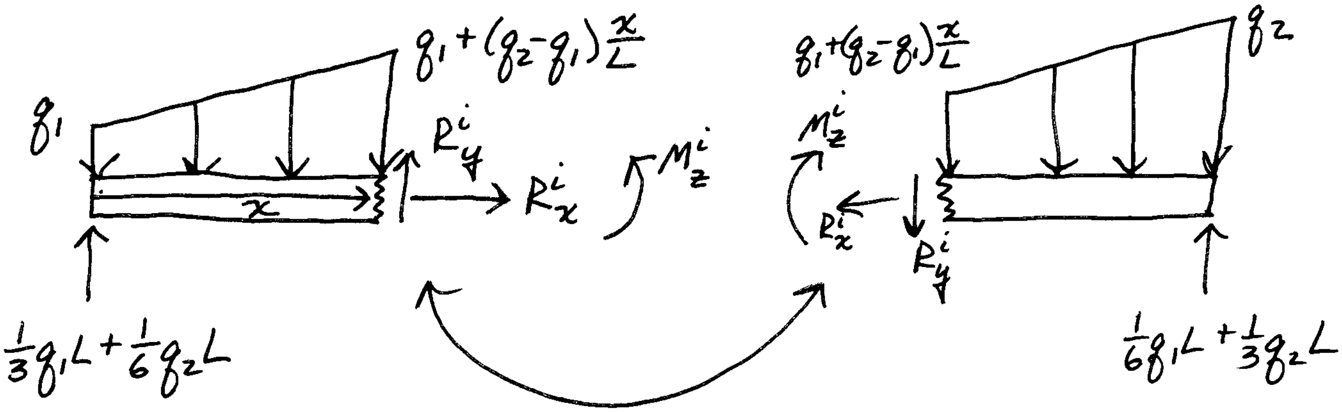
$$\Sigma M_z^{pt.A} = R_y^B L - q_1 L \frac{L}{2} - \frac{1}{2}(q_2 - q_1)L \frac{2}{3}L = 0$$

Solve ~~for~~ R_y^A & R_y^B $\rightarrow$

$$\boxed{\begin{aligned} R_y^A &= \frac{1}{3}q_1 L + \frac{1}{6}q_2 L \\ R_y^B &= \frac{1}{6}q_1 L + \frac{1}{3}q_2 L \end{aligned}}$$

Now, to analyze the internal ~~forces~~ we must cut **both** the beam and the distributed load.

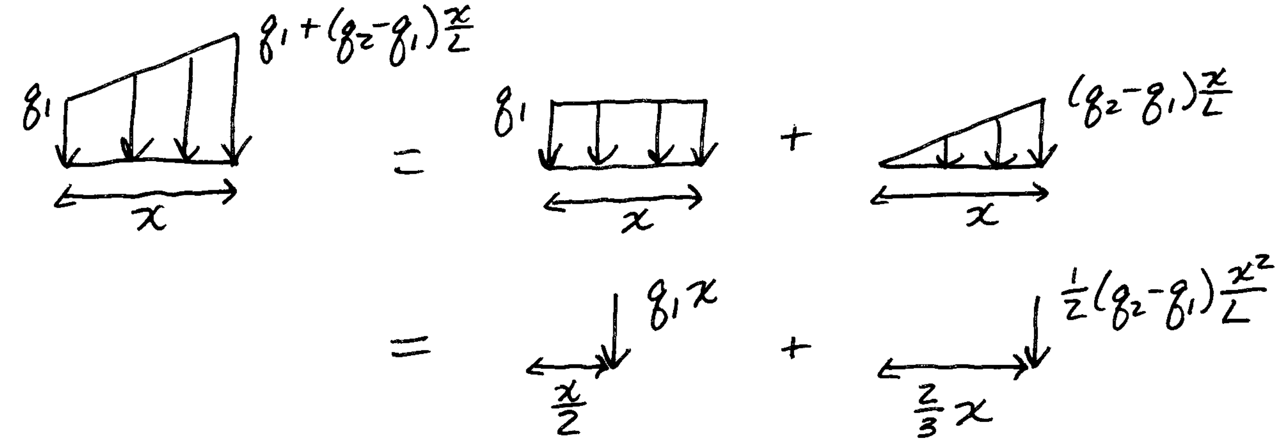
Redraw the FBD after the cut at x



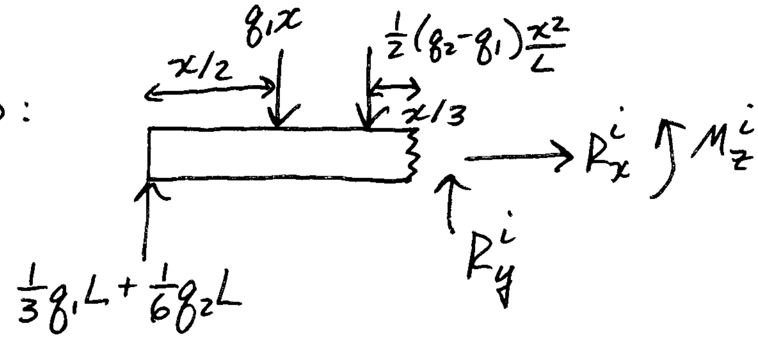
Note equal and opposite internal forces.

A beam can internally support an axial force, R_x^i, a shear force, R_y^i, and an internal couple (usually called an internal moment) M_z^i.

Now we can re-analyze the cut distributed load.



Re-draw ~~our~~ FBD:



Now analyze the equilibrium equations.

$$\Sigma F_x = \boxed{R_x^i = 0}$$

$$\Sigma F_y = R_y^i + \frac{1}{3}q_1 L + \frac{1}{6}q_2 L - q_1 x - \frac{1}{2}(q_2 - q_1)\frac{x^2}{L} = 0$$

$$\therefore \boxed{R_y^i = (q_2 - q_1)\frac{x^2}{2L} + q_1 x - \frac{1}{3}q_1 L - \frac{1}{6}q_2 L}$$

Notice at $x = 0$ $R_y^i = -R_y^A$
and at $x = L$ $R_y^i = R_y^B$.

You should convince yourself that this makes sense by looking at the left and right FBDs.

$$\Sigma M_z^{\text{about point } x} = M_z^i - \frac{1}{3}q_1 L x - \frac{1}{6}q_2 L x + q_1\frac{x^2}{2} + \frac{1}{2}(q_2 - q_1)\frac{x^2}{L}\frac{x}{3} = 0$$

$$\therefore \boxed{M_z^i = -(q_2 - q_1)\frac{x^3}{6L} - q_1\frac{x^2}{2} + \frac{1}{3}q_1 L x + \frac{1}{6}q_2 L x}$$

You should convince yourself that the dimensions of each term in this equation are correct.

Now, to find the maximum internal moment we set

$$\frac{dM_z^i}{dx} = 0 \text{ and solve for } x.$$

$$\frac{dM_z^i}{dx} = \underbrace{-(q_2-q_1)\frac{x^2}{2L} - q_1 x + \frac{1}{3}q_1 L + \frac{1}{6}q_2 L}_{\text{Notice this equals } -R_y^i} = 0$$

$$x = \frac{q_1 \pm \sqrt{q_1^2 + 4(q_2-q_1)\frac{1}{2L}\left(\frac{1}{3}q_1 L + \frac{1}{6}q_2 L\right)}}{-\frac{1}{L}(q_2-q_1)}$$

$$x = L\,\frac{q_1 \pm \sqrt{q_1^2 + q_1 q_2 + q_2^2}}{\sqrt{3}\,(q_1 - q_2)}$$

Notice this equation breaks down for $q_1 = q_2$. In that special case our original equation simplifies and we would get

$$-q_1 x + \frac{1}{3}q_1 L + \frac{1}{6}q_1 L = 0 \rightarrow x = \frac{L}{2} \checkmark$$

For cases with $q_2 > q_1$ we want the $-$ root.

i.e. $$x = L\,\frac{q_1 - \sqrt{q_1^2 + q_1 q_2 + q_2^2}}{\sqrt{3}\,(q_1 - q_2)}$$

Take the specific example of $q_1 = 0$, then

$x_{crit} = L/\sqrt{3}$ → greater than $\frac{L}{2}$ but less than $\frac{L}{3}$.

$$\rightarrow M_z^{max} = -\frac{1}{18\sqrt{3}}q_2 L^2 + \frac{1}{6\sqrt{3}}q_2 L^2 = \frac{1}{9\sqrt{3}}q_2 L^2$$