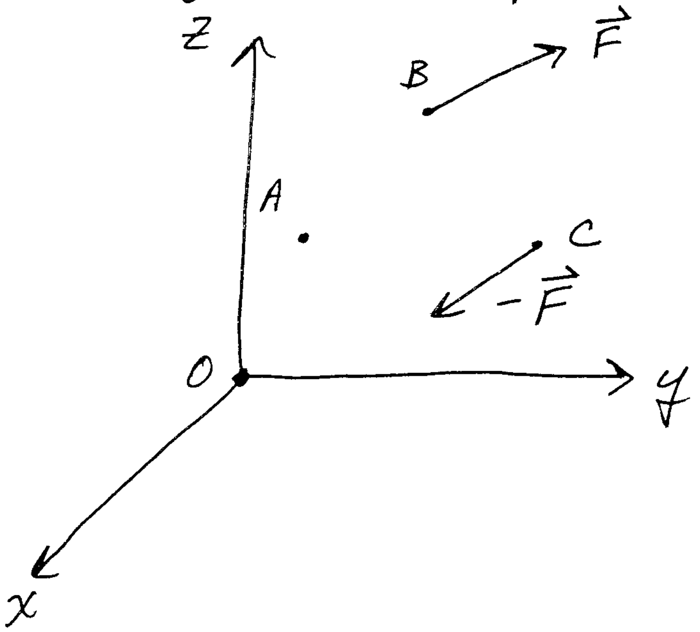
Couples

Two equal and opposite non-colinear forces.



Recall: $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$

What is the moment due to the forces about some arbitrary point A ?

$$\vec{M}_+ = \vec{r}_{B/A} \times \vec{F}$$

$$\vec{M}_- = \vec{r}_{C/A} \times (-\vec{F})$$

$$\vec{C}_A = \vec{M}_+ + \vec{M}_- = \vec{r}_{B/A} \times \vec{F} - \vec{r}_{C/A} \times F$$

$$= (\vec{r}_{B/A} - \vec{r}_{C/A}) \times \vec{F}$$

$$= (\vec{r}_B - \vec{r}_A - \vec{r}_C + \vec{r}_A) \times \vec{F}$$

$$= (\vec{r}_B - \vec{r}_C) \times \vec{F}$$

$$\vec{C} = \vec{r}_{B/C} \times \vec{F}$$

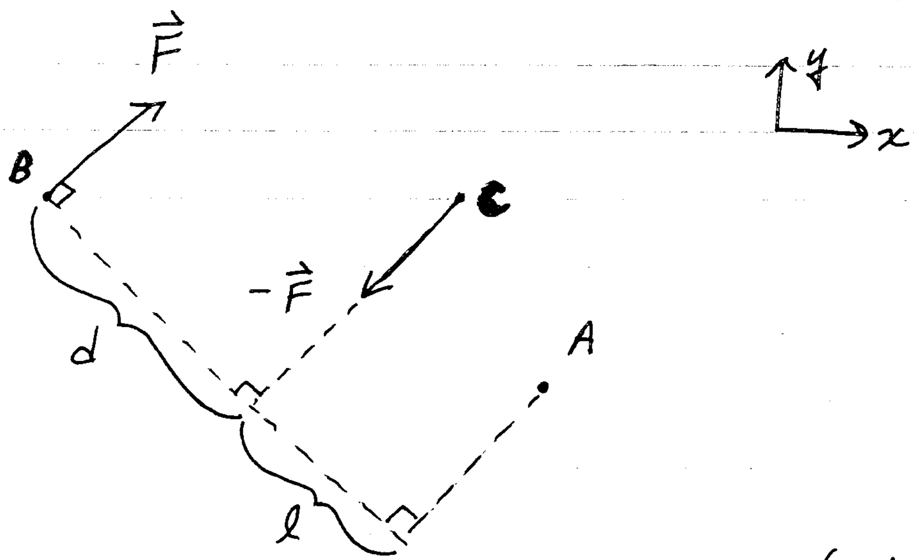
Note these expressions do not depend on $\vec{r}_A$ in any way!

$\therefore$ The moment due to a couple is the same no matter what points the moments are taken about.

Note, furthermore that the sum of the forces due to a couple are zero.

i.e. $\vec{R} = \vec{F} + -\vec{F} = 0$

2-D Scalar Representation



$$\vec{M}_A^{+} = -F(d+\ell)\vec{k}$$

$$\vec{M}_A^{-} = F(\ell)\vec{k}$$

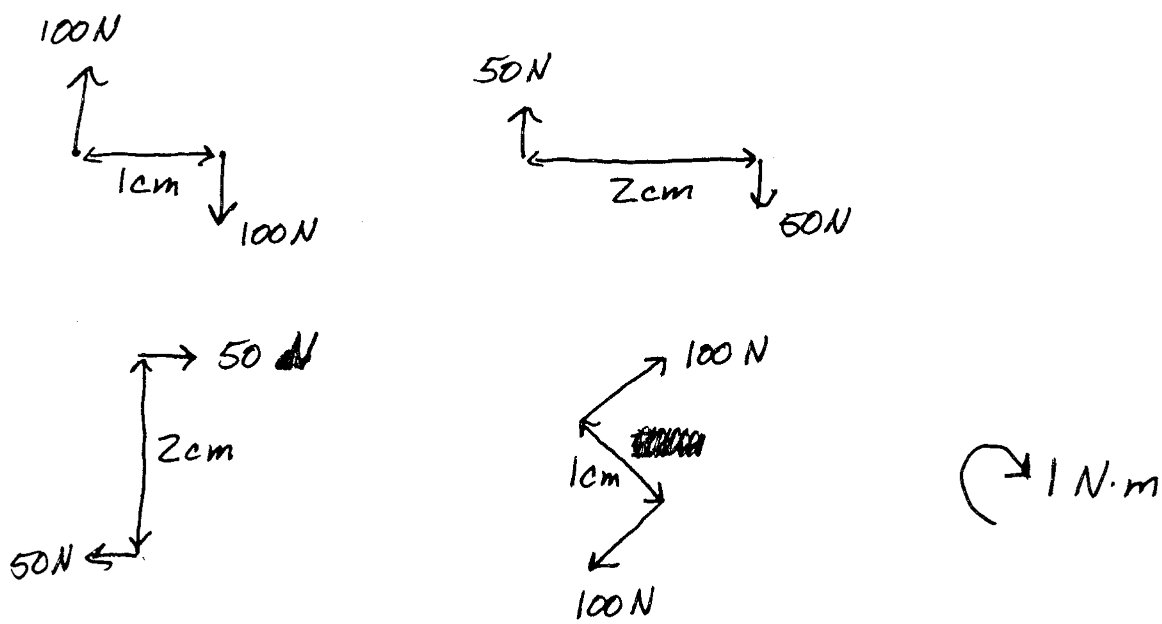
$$\vec{C} = \vec{M}_A^{+} + \vec{M}_A^{-} = -Fd\,\vec{k}$$

(Note the force couple causes a clockwise ↻ i.e. $-\vec{k}$ rotation)

Again $\vec{C}$ did not depend on the location of A in any way, hence the couple about <u>any</u> point is always the same.

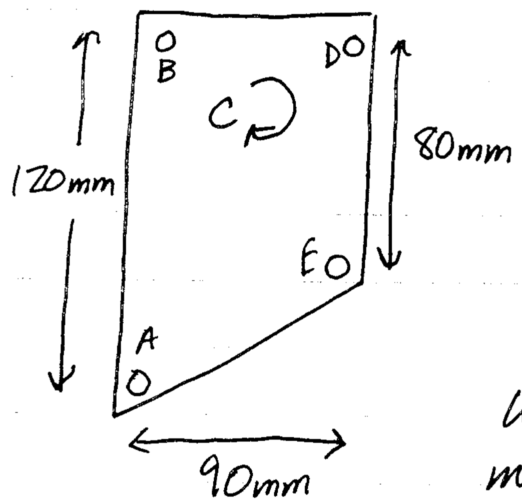
This also means that conceptually you can move a couple anywhere in space and it will have the same effects on a rigid body.

<u>Equivalent Couples</u>



Sometimes we also use the "moment arrow" i.e. $\twoheadrightarrow$ for 3-D drawings.

Example Problem 2.75



You have 2 bolts to place in 2 of the 4 holes. Where should you place the bolts to minimize the shearing force on them? What is the max C if the max shearing force on the bolts can be 40N?

$\Sigma \vec{M} = -C\vec{k} + Fd\vec{k}$ where Fd is the moment applied ~~by~~ the bolts.

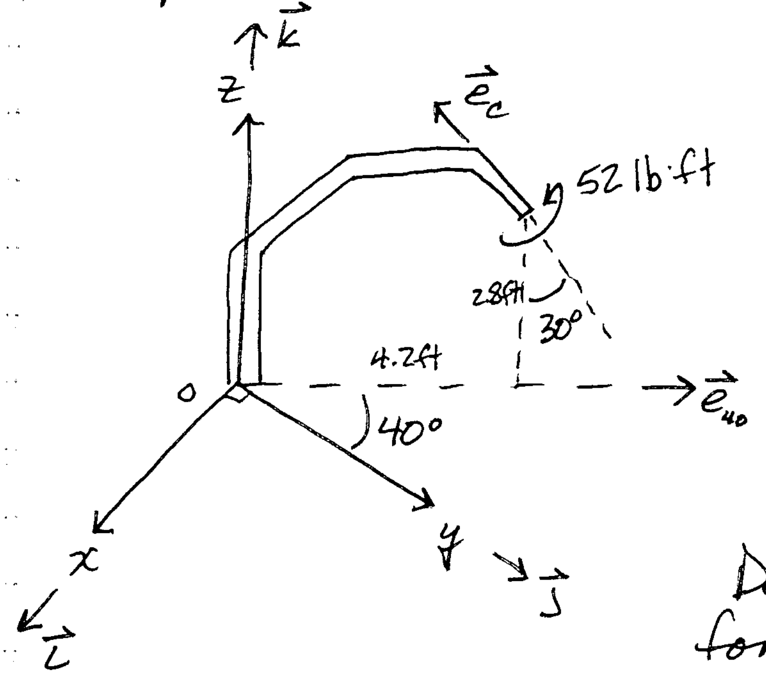
$\therefore F = \frac{C}{d}$ → to minimize F we must maximize d.

Therefore we want to place bolts in holes A and D.

The distance d between A and D is $\sqrt{120^2+90^2}$ $= 150$ mm.

$\therefore C_{max} = F_{max} \cdot d = 40\,N \cdot 150\,mm = 6\,N \cdot m$

Example Problem 2.79

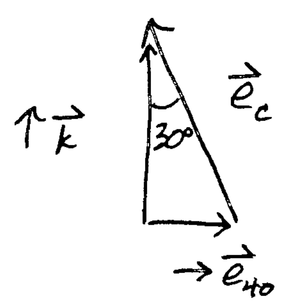


For clarity I have drawn the robot arm in the z-$\vec{e}_{40}$ plane and this is the plane of the page. The $\vec{e}_{40}$ direction is in the x-y plane. Determine the vector form of the couple and the moment of the couple about the z-axis.

$$\vec{C} = C\,\vec{e}_c$$

C = magnitude of the couple
$\vec{e}_c$ = direction (unit vector)

In the z - $\vec{e}_{40}$ plane we have

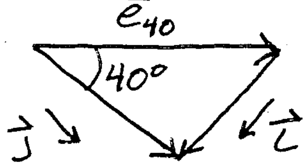


$$\vec{e}_c = \underbrace{(\vec{e}_c \cdot \vec{e}_{40})}_{-\sin 30^\circ}\,\vec{e}_{40} + \underbrace{(\vec{e}_c \cdot \vec{k})}_{\cos 30^\circ}\,\vec{k}$$

— minus b/c $\vec{e}_c$ opposes $\vec{e}_{40}$

$$\therefore\ \vec{e}_c = -\frac{1}{2}\,\vec{e}_{40} + \frac{\sqrt{3}}{2}\,\vec{k}$$

Now consider $\vec{e}_{40}$ in the x-y plane



$$\vec{e}_{40} = \underbrace{(\vec{e}_{40} \cdot \vec{i})}_{-\sin 40^\circ}\vec{i} + \underbrace{(\vec{e}_{40} \cdot \vec{j})}_{\cos 40^\circ}\vec{j}$$

$$\therefore \vec{e}_{40} = -\sin 40^\circ \vec{i} + \cos 40^\circ \vec{j}$$

$$\therefore \vec{e}_c = \tfrac{1}{2}\sin 40^\circ \vec{i} - \tfrac{1}{2}\cos 40^\circ \vec{j} + \tfrac{\sqrt{3}}{2}\vec{k}$$

$$\therefore \vec{C} = 52 \text{ lb}\cdot\text{ft} \left(\tfrac{1}{2}\sin 40^\circ \vec{i} - \tfrac{1}{2}\cos 40^\circ \vec{j} + \tfrac{\sqrt{3}}{2}\vec{k}\right)$$

Moment due to the couple about the z-axis.

Recall $\vec{M}_{axis} = (\vec{M}_{pt} \cdot \vec{e}_{axis})\vec{e}_{axis}$

where $\vec{M}_{pt}$ is the moment about any point on the axis. But, the moment due to $\vec{C}$ about <u>every</u> point is just $\vec{C}$.

$$\therefore \vec{M}_{z\text{-axis}} = (\vec{C} \cdot \vec{k})\vec{k}$$

$$= 52\left(\tfrac{\sqrt{3}}{2}\right)\vec{k} \text{ ft}\cdot\text{lb}.$$

Equivalent Systems

Two sets of force systems are equivalent (as far as their effects on a rigid body are concerned) if the sum of all of the forces and the sum of the moments about <u>any</u> point in space are identical.

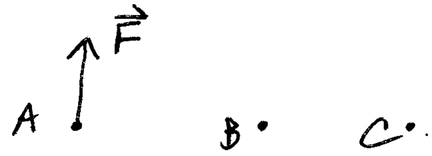
$$\Sigma \vec{F} = \sum_{i=1}^{N} \vec{F}^{(i)}$$

$$\Sigma \vec{M}_A = \sum_{i=1}^{M} \vec{C}^{(i)} + \sum_{i=1}^{N} \vec{r}_{i/A} \times \vec{F}^{(i)}$$

where there are N forces acting at the positions $\vec{r}_i$ and there are M arbitrarily positioned couples.

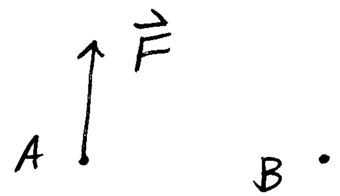
Recall that the net force of a couple is zero and the moment due to a couple about every point in space is identical. Hence, a couple can be moved to any point in space without changing its effects on a rigid body.

However, changing the position of a force does change its moment about points in space.

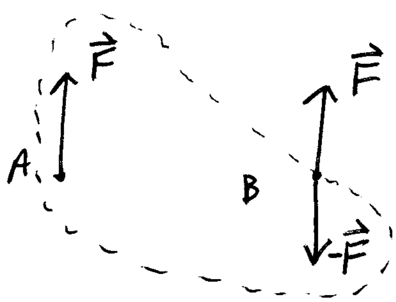


For example, if I move the force to point B its moment about point C decreases and its moment about point B vanishes.

So if we insist on moving a force we must compensate for this change in moment. Here is how we do this.



We want to move $\vec{F}$ from A to B.

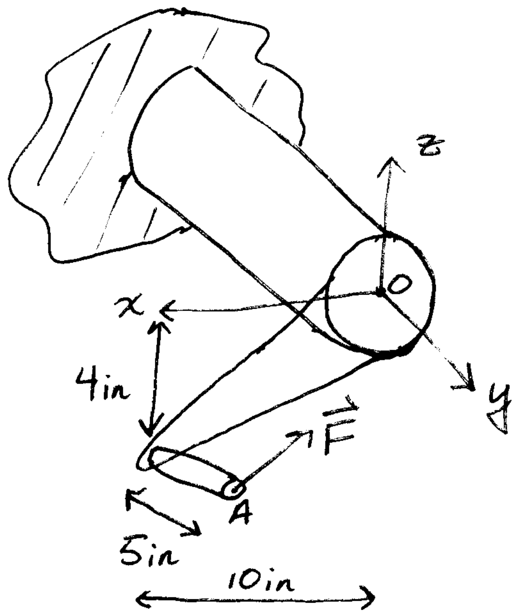


At point B, place both $\vec{F}$ and its opposite $-\vec{F}$.

Now the $\vec{F}$ and $-\vec{F}$ surrounded by the dotted line constitute a couple $\vec{C} = \vec{r}_{A/B} \times \vec{F}$.

What's left is $\vec{F}$ acting at point B and the couple $\vec{C}$ floating around, i.e. acting at any point you like.

Example Problem 2.91



$\vec{F} = 2500\vec{\imath} + 4000\vec{\jmath} + 3000\vec{k}$ lb.

Move $\vec{F}$ to O.
Resolve this force into a normal and shearing component.
Resolve the couple into a twisting and bending component.

By moving $\vec{F}$ to O we must also add a couple $\vec{C}$ to compensate for the change in moment.

$$\vec{C} = \vec{r}_{A/O} \times \vec{F}$$

$$\vec{r}_{A/O} = \vec{r}_A = 10\vec{\imath} + 5\vec{\jmath} - 4\vec{k} \text{ in}$$

$$\vec{r}_{A/O} \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 10 & 5 & -4 \\ 2500 & 4000 & 3000 \end{vmatrix} = 31000\vec{\imath} - 40000\vec{\jmath} + 27500\vec{k} \text{ lb}\cdot\text{in}$$

$$\therefore \vec{C} = 31000\vec{\imath} - 40000\vec{\jmath} + 27500\vec{k} \text{ in}\cdot\text{lb}$$

The normal force is that parallel to the y-axis
i.e. $\vec{F}_N = 4000\vec{\jmath}$ lbs.

Then the shearing part of the force is the remainder.

i.e. $\vec{F}_s = \vec{F} - \vec{F}_N = 2500\vec{i} + 3000\vec{k}$ lbs.

The twisting component of $\vec{C}$ is parallel to $\vec{j}$, i.e.
$$\vec{C}_T = (\vec{C} \cdot \vec{j})\vec{j} = -40000\vec{j} \text{ in·lb}$$

The remainder is the bending component of $\vec{C}$, i.e.
$$\vec{C}_B = \vec{C} - \vec{C}_T = 31000\vec{i} + 27500\vec{k} \text{ in·lb.}$$