

Math211 2004 Spring Project

Pumping a playground swing

The goal of the project is to explain why (and how) kids can pump a swing on the playground, so parents can sit back after giving an initial push. Kids basically change the length of the swing by squatting down and standing up (or moving their legs forward and backward). The name of this phenomenon is *parametric resonance*. We will examine this physical system through a number of steps and use quantitative, qualitative and numerical methods. The differential equation which describes the motion of a mathematical pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0,$$

where θ is the angular displacement of the pendulum from the vertical, L is the length of the pendulum arm and g is the acceleration due to gravity. The motion of a playground swing follows a similar equation, we only have to add a term which describes the effect of damping:

$$\frac{d^2\theta}{dt^2} + c \frac{d\theta}{dt} + \frac{g}{L} \sin \theta = 0,$$

where $c > 0$ and depends on how well oiled the swing's hinges are.

1. Write the differential equation as a system of two first order differential equation by letting $x = \theta$ and $y = \theta'$. Then examine numerically the solutions of this system by choosing $g/L = 5$ and $c = 0.2$. Concentrate on the behavior of x , which is the angular displacement from the vertical as a function of time. Choose an initial condition fairly close to the equilibrium $x = y = 0$ and a time interval which is long enough to see the characteristic behavior. (You will find Chapter 8 of the manual very useful.) What can you say about the stability of the zero solution? What can you say if we choose $c = 0$? (Really well-oiled hinges.)
2. Now let's examine the effect of pumping. The pumping changes the length of the swing. Also, the pumping is a periodic function of time. Do again a numerical experiment, but choose now $g/L = 5 + \cos(\alpha t)$ and $c = 0.2$. Let α vary from 3 to 5 by increments of 0.5. (Choose the same initial conditions and time interval as before.) Concentrate again on the behavior of x . What can you say about the stability of the zero solution? How is this in accordance with your experience?

For small displacements of the swing the value of θ is small, and we can make the simplifying assumption $\sin \theta \approx \theta$.

3. Consider now the equation $\theta'' + \omega^2(t)\theta = 0$ where $\omega(t)$ is a periodic function with period T , i.e. $\omega(t) = \omega(t + T)$. Let $x = \theta$ and $y = \theta'$ again. Prove now that if $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution of the resulting system, then $\begin{bmatrix} x(t + T) \\ y(t + T) \end{bmatrix}$ is also a solution.

The next part is some necessary background material. You will need these for parts 4. and 5. later.

The previous result implies that we can create a *map* of the phase plane \mathbb{R}^2 in the following way: to each $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ we can assign the point $\begin{bmatrix} x(T) \\ y(T) \end{bmatrix}$, where $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution of the IVP of the above system of differential equations with $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. For example, the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ will be mapped into $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because if we start a solution at the origin, then the solution of the IVP is just the constant $x(t) \equiv 0$, $y(t) \equiv 0$ and then of course $\begin{bmatrix} x(T) \\ y(T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This map is called the *Poincaré map*. Looking at the consecutive images of a certain point we look at the values $\begin{bmatrix} x(nT) \\ y(nT) \end{bmatrix}$, so we can draw conclusions about the long-term behavior of a solution.

We will use the following theorem. (The proof can be found for example in Arnold, V.I.: *Ordinary Differential Equations*, MIT Press, 1978.)

Theorem. The above map is a linear map and if it is given by the 2×2 matrix A , then $\det(A) = 1$. The origin is stable if $|\text{Tr}(A)| < 2$ and unstable if $|\text{Tr}(A)| > 2$.

A linear map is just a multiplication by a constant matrix, so if it is given by A , then $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ will be mapped into $A \begin{bmatrix} x \\ y \end{bmatrix}$.

The columns of the matrix A are given by the images of the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

4. We will compute now the above-mentioned matrix A for the following system:

$$\theta'' + \omega^2(t)\theta = 0,$$

where $\omega(t)$ is a periodic function with period 2π and is given by

$$\omega(t) = \begin{cases} \omega + \varepsilon & 0 \leq t < \pi \\ \omega - \varepsilon & \pi \leq t < 2\pi \end{cases}$$

(ω, ε are constants.) We will suppose that ε is small compared to ω , so we change the length of the swing only a little bit. Compute now A by using the fact that $A = A_2 A_1$, where A_1 is the matrix of the above linear map using $\omega + \varepsilon$ in the differential equation and evaluating the solutions at $T = \pi$ and A_2 is the matrix of the above linear map using $\omega - \varepsilon$ in the differential equation and evaluating the solutions at $T = \pi$ again. (This fact is a consequence of the linearity of the map.)

5. Write down the equation $|\text{Tr}(A)| = 2$. This specifies the boundary between the regions of stability and instability on the $\omega - \varepsilon$ plane. Use the MATLAB code `implplot.m` to draw a graph of this boundary. Use the ranges $(0, 2.2)$ for ω and $(-0.3, 0.3)$ for ε . (The code can be downloaded from the homepage of the course.) Specify the regions of stability and instability of the origin. How is this in accordance with your experiences?