In general, take off 1 pt for arithmetic mistakes, with max of 1 pt off per portion of problem (a.) or (b.), etc.)

Math 211 Fall 1999 Exam 2: Part 1

Tuesday, November 9, 1999

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **75 minutes** to complete this exam. Do all 5 problems. The points for this part total to 60. Please do all of your work in the blue books. Remember to write and sign the pledge on your blue book. **Please put your name, your Rice ID number, and the name of your instructor on the front of each of your blue books.**

1. Consider the two by two matrices

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

   (a) (4pts) Calculate AB.

   (b) (4pts) Calculate BA.

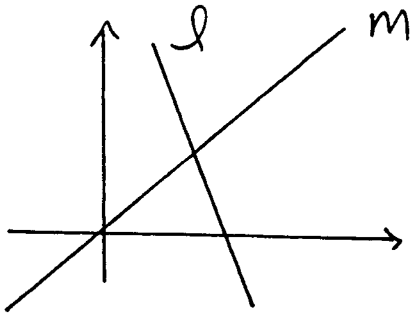
   (c) (2pts) What does this show about the commutativity of matrix multiplication?

a. $AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$

b. $BA = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$

c. Since $BA \neq AB$, matrix multiplication is not commutative.

2. Consider the following graph.



(a) (2pts) Is it possible that line ℓ is the solution set of an inhomogeneous system of equations and m is the solution set of the corresponding homogeneous system?

(b) (4pts) Why or why not?

a. NO

b. Because the solutions to the homogeneous ~~sa~~ eqn and the inhomog. eqn cannot intersect

or

because the soln to hom & inhom should be parallel

or

because the soln $\vec{x}$ to the inhom. eqn has the form

$$\vec{x} = \vec{x}_p + \vec{x}_h$$

where $\vec{x}_p$ is a particular soln to inhom. system and $\vec{x}_h$ is the gen. soln to hom. eqn

If they write the correct answer plus some incorrect statements, take off one pt.

3. Consider a reaction between two kinds of chemicals, modeled by the system of equations

$$
\begin{aligned}
x' &= (y-2)(x-2) \\
y' &= -y^2 + 4y - 3
\end{aligned}
$$

where $x(t)$ is the amount of the chemical X-onium, and $y(t)$ is the amount of chemical Y-onite.

(a) (8pts) Plot the nullclines for each equation. Use a solid line for the nullcline corresponding to the first equation, and a dashed line for the second equation.

(b) (4pts) Calculate the coordinates of all equilibrium points of the system. Plot the points on your sketch from part 3a and label them with their coordinates.

(c) (6pts) Suppose that if the amount of X-onium, which is poisonous, ever becomes greater than 3 units, the chemist performing the experiment will die. If the chemist starts the reaction with 1 unit of X-onium and 2 units of Y-onite, will the chemist live or die? Justify your answer.

(vertical)

a. X nullclines $\quad x' = (y-2)(x-2) = 0 \quad$ (see graph)

2pts $y = 2$ and 2pts $x = 2$ are nullclines

(horizontal)

Y nullclines $\quad y' = -y^2 + 4y - 3 = -(y-1)(y-3) = 0$

2pts $y = 1$ and 2pts $y = 3$ are nullclines

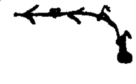
b. equilibria: where an X nullcline meets a Y nullcline

2pts $(2,1)$ $\quad$ $(2,3)$ 2pts

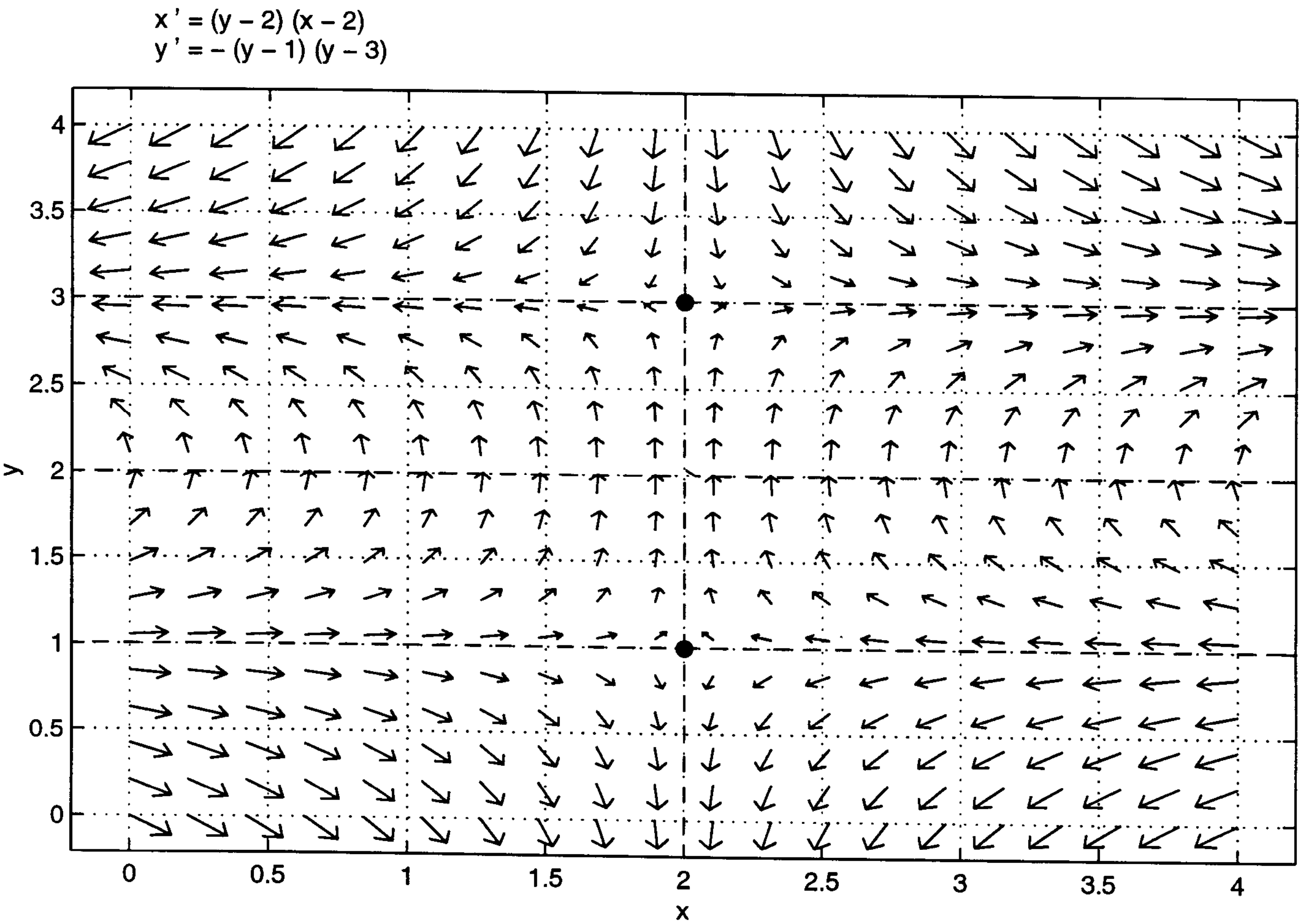
c. The chemist will live. 2pts $\quad$ 4 pts for reason

Because the line $x = 2$ is a vertical nullcline, any solution starting in the region with $x < 2$ must remain in that region and not hit 3.

OR By looking at direction field, can trace out trajectory and see that trajectory looks like this.



$(1,2)$



take off one point if they think that they have to check if y can be > 3.

4. Consider a spring with damping constant $\mu = 12$ and spring constant $k = 18$. Hang a ball of mass m on the end of the spring, and consider the motion of the spring.

   (a) (2pts) Write down the differential equation corresponding to the motion of the spring.

   (b) (4pts) Describe the m for which the motion of the spring will be overdamped, the m for which it will be critically damped, and the m for which it will be underdamped.

   (c) (3pts) **For parts (c), (d), and (e), assume that m=3.** Find the general solution for the differential equation in part 4a.

   (d) (3pts) Suppose the spring is initially stretched to distance .1 meters and released with initial velocity equal to 0. Find the particular solution corresponding to these initial conditions.

   (e) (2pts) Describe in a few words the physical behavior of the spring after it is released.

a. $y'' + \frac{12}{m}y' + \frac{18}{m}y = 0$ one pt for $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0$

or $my'' + 12y' + 18y = 0$

b. char. eqn

$$\lambda^2 + \frac{12}{m}\lambda + \frac{18}{m} = 0$$

If $\left(\frac{12}{m}\right)^2 - 4\cdot\left(\frac{18}{m}\right)$ is > 0 overdamped, $= 0$ crit damped, < 0 underdamped } 2 pts if they know what they have to calculate

calculating we see

$$\frac{144}{m^2} - \frac{72}{m} > 0$$

$$\frac{144}{m^2} > \frac{72}{m}$$

$144 > 72m$

2 pts this part {

$$\boxed{\begin{array}{ll} 2 > m & \text{overdamped} \\ 2 = m & \text{crit. damped} \\ 2 < m & \text{underdamped} \end{array}}$$

−1 if they mix up over & under

c. $m = 3$ under damped $\quad y'' + 4y' + 6y = 0$

$\lambda^2 + 4\lambda + 6 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16-24}}{2}$$

1 pt for this line

$\lambda = -2 \pm \sqrt{2}\, i = a \pm bi \quad$ so $\quad a = -2 \quad b = \sqrt{2}$

gen soln:

$$y(t) = e^{at}\left(A_1 \cos(bt) + A_2 \sin(bt)\right)$$ ← 2 pts if they get this

$$\boxed{y(t) = e^{-2t}\left(A_1 \cos(\sqrt{2}\, t) + A_2 \sin(\sqrt{2}\, t)\right)}$$

d. $y(0) = A_1 = .1$ ← 1 pt

$$y'(t) = -2e^{-2t}\left(A_1 \cos(\sqrt{2}\, t) + A_2 \sin(\sqrt{2}\, t)\right) + e^{-2t}\left(-\sqrt{2} A_1 \sin(\sqrt{2}\, t) + \sqrt{2} A_2 \cos(\sqrt{2}\, t)\right)$$

$$y'(0) = -2A_1 + \sqrt{2} A_2 = 0$$

$$-.2 + \sqrt{2} A_2 = 0$$

$$\sqrt{2} A_2 = .2$$

$$A_2 = \frac{.2}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{10}$$ 1 pt

$$\boxed{y(t) = e^{-2t}\left(.1 \cos(\sqrt{2}\, t) + \frac{\sqrt{2}}{10} \sin(\sqrt{2}\, t)\right)}$$ 1 pt

e. the spring <u>oscillates</u> 2 pts

5. (a) (2pts) Write the following system of equations in matrix form $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x - 4y + z &= 6 \\ 3x - 4y + 2z &= 5 \\ -6x + 8y - 4z &= -10. \end{aligned}$$

(b) (6pts) Find the solution space of the system in part 5a. Put your answer in parametrized form.

(c) (4pts) Is the matrix A for this system singular or nonsingular? Justify your answer.

a. $\begin{bmatrix} 2 & -4 & 1 \\ 3 & -4 & 2 \\ -6 & 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -10 \end{bmatrix}$

or

$A = \begin{bmatrix} 2 & -4 & 1 \\ 3 & -4 & 2 \\ -6 & 8 & -4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 6 \\ 5 \\ -10 \end{bmatrix}$

b. augmented matrix

$\begin{bmatrix} 2 & -4 & 1 & 6 \\ 3 & -4 & 2 & 5 \\ -6 & -8 & -4 & -10 \end{bmatrix}$ 1 pt for this only

row reduce to row ech. form : $\begin{bmatrix} 2 & -4 & 1 & 6 \\ 0 & 2 & 1/2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (one possible answer, beware of equivalent answers)

4 pts if they got this for

or to reduced row ech form: $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1/4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

then backsolve
z is a free variable

1 pt for knowing z is free (or writing $z=t$)

so we get

$$y + \tfrac{1}{4}z = -2$$

$$y = -2 - \tfrac{1}{4}z$$

$$x + z = -1$$

$$x = -1 - z$$

soln space is

$$\begin{pmatrix} -1 - z \\ -2 - \tfrac{1}{4}z \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -\tfrac{1}{4} \\ 1 \end{pmatrix}$$

$$\text{or } \boxed{\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -\tfrac{1}{4} \\ 1 \end{pmatrix}}$$

beware of equivalent answers

1 pt

c. $\boxed{\text{Singular}}$ because
There exists more than one solution to the eqn $A\vec{x} = \begin{pmatrix} 6 \\ 5 \\ -10 \end{pmatrix}$

OR In the reduced row ech form or row ech form of A, there is a zero on the diagonal.

3 pts for reason

Again, if they write a correct reason + some incorrect stuff, take off one pt.