

Math 211 Fall 1999 Exam 1: Part 1 SOLUTIONS

Tuesday, October 5, 1999

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **75 minutes**. Do all 5 problems. The points for this part total to 60. Please do all your work in the blue books. Remember to write and sign the pledge on your blue book. **Please put your name and the name of your instructor on the front of each of your blue books.**

1. [10] Solve the initial value problem

$$\frac{dx}{dt} = \frac{t^2}{x \cos(x^2 + \pi)}, \quad x(-1) = 0.$$

You may leave your answer in implicit form.

Separable equation:

$$\begin{aligned} x \cos(x^2 + \pi) dx &= t^2 dt \\ \int x \cos(x^2 + \pi) dx &= \int t^2 dt \\ (1/2) \sin(x^2 + \pi) &= t^3/3 + C. \end{aligned}$$

Solve for the constant:

$$(1/2) \sin(0 + \pi) = 0 = -1/3 + C \Rightarrow C = 1/3.$$

Hence $\boxed{\sin(x^2 + \pi) = \frac{2}{3}(t^3 + 1)}.$

If they get an explicit solution:

$$x = \pm \sqrt{\sin^{-1}(2(t^3 + 1)/3) - \pi}.$$

2. [10] Consider the differential equation

$$ty' = 2y + 2t^3$$

with the initial condition $y(1) = 0$.

- (a) Find the particular solution for this ordinary differential equation.
 - (b) Explain why no solution exists for the initial condition $y(0) = 3$.
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(a) In normal form,

$$y' - (2/t)y = 2t^2.$$

Integrating factor:

$$u(t) = \exp\left(\int (-2/t) dt\right) = \exp(-2 \ln t) = t^{-2}.$$

Hence

$$\begin{aligned}(t^{-2}y)' &= t^{-2} \cdot 2t^2 \\(t^{-2}y)' &= 2 \\t^{-2}y &= 2t + C \\y &= 2t^3 + Ct^2.\end{aligned}$$

Solve for C :

$$0 = 2 + C \Rightarrow C = -2.$$

So $y = 2t^3 - 2t^2$.

- (b) Substituting in $t = 0$ and $y = 3$ gives a contradiction in the initial equation:

$$0 = 6 + 0!!!$$

Hence there is no solution with this initial condition.

Also, one can note that there is no way to solve for the constant C in the method used in part (a):

$$3 = 0 + C * 0 = 0$$

has no solution for C . Strictly speaking this only shows that this particular *method* fails, but it does strongly suggest that there is no solution.

3. [10] Suppose $y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = (y^2 - 4) \cdot \ln(t^2 + 1) \text{ and } y(3) = 1.$$

Show that $-2 < y(t) < 2$ for all t for which y is defined.

First note that the constant functions $y(t) = 2$ and $y(t) = -2$ are solutions to the equation:

$$\frac{dy}{dt} = 0 = (2^2 - 4) \cdot \ln(t^2 + 1).$$

Since the function on the right hand side of the equation is a smooth function of y and t , the uniqueness theorem holds. Hence solution curves cannot cross. Since the solution to the initial value problem with $y(3) = 1$ starts out between the horizontal solution curves $y = \pm 2$, it must stay between them for as long as the solution is defined, i.e. $-2 < y(t) < 2$.

(*Aside:* One can note that in fact the existence theorem shows that the solution, since it is bounded in y , must exist for all time t .)

4. [15] We are familiar with modeling a population using the logistic equation. In this problem we will see that the same qualitative behavior appears for a large class of different models.

Suppose we are modeling a population with a differential equation of the form

$$P' = a(P) \cdot P \tag{1}$$

where $a(P)$ is a function satisfying

$$a(0) > 0;$$

a is a strictly decreasing function of P ;

$a(B) = 0$, where B is a positive number.

- Find the equilibrium solutions of equation (1). (*Hint*: remember that the right hand side of equation (1) is $a(P) \cdot P$, not $a(P)$.)
- Classify these equilibria as to stability.
- Sketch the solution curves of the system, with the equilibrium solutions clearly indicated. Be sure to draw at least one solution in each region between the equilibrium solutions.
- Compare your answers to the case of the logistic equation. Explain the physical significance of the quantity B .

(a) Since a is a strictly decreasing function, its only zero is at B . Hence the only zeros of $f(P) = a(P) \cdot P$ are $P = 0$ and $P = B$. These are the two equilibrium solutions of the equation.

(b) We just need to know the behavior of the function $f(P)$ near $P = 0$ and $P = B$. Since $a(0)$ is positive, a will be positive for small values of P . Hence $f(P)$ will be negative for small negative P , and positive for small positive P . This implies that 0 will be an **unstable** equilibrium for the differential equation.

We analyze the equilibrium $P = B$ similarly. Here $a(P)$ is positive for P slightly less than B , and $a(P)$ is negative for P slightly greater than B . Since P is a positive number on either side of the equilibrium, the function $f(P)$ has the same sign as $a(P)$. This implies that the equilibrium $P = B$ is **stable**.

(c) See attached figure.

(d) The qualitative behavior of this system is exactly the same as in the logistic equation, which is the special case $a(P) = r(1 - P/K)$. This shows that the special form of $a(P)$ in the logistic case was not crucial, only the properties listed above. By comparison to the logistic case, or by looking at the solution behavior, we can reasonably call B the **carrying capacity** for this population.

5. [15] You have made a pot of 3 liters of chili for a party, but you realize that it is too spicy. The chili you have made is 4% chili powder by volume, but you want it to be 1% chili powder. To fix this, as you remove chili from the pot at a rate of 1 liter per minute, you add more tomatoes and beans (no chili powder) to the pot at a rate of 2 liters per minute. (Assume perfect mixing.) At what time will the chili have your desired concentration of 1% chili powder?

Variables:

- t = time, in minutes;
- $V(t)$ = total volume in the pot, in liters;
- $a(t)$ = total amount of chili powder in the pot, in liters;
- $c(t)$ = concentration of chili powder in the pot.

Here the concentration and the amount are related by

$$c(t) = \frac{a(t)}{V(t)}.$$

We will get an ODE for $a(t)$ and then convert to $c(t)$ at the end.

Since stuff is flowing in at 2 liters/min and out at 1 liter/min, we have

$$V(t) = 3 + (2 - 1)t = 3 + t.$$

We obtain the differential equation for $a(t)$ by considering the rate of flow in and out. There is no chili powder flowing in, so

$$a'(t) = -\text{rate out} = -1\text{liter}/\text{min} * c(t) = -\frac{a(t)}{V(t)}$$

i.e.

$$a'(t) = -\frac{a(t)}{3+t}.$$

This is a linear equation, and the integrating factor is

$$u(t) = \exp\left(\int dt/(3+t)\right) = 3+t.$$

Hence

$$\begin{aligned}(a(t)(3+t))' &= 0 \\ a(t)(3+t) &= C \\ a(t) &= \frac{C}{3+t}.\end{aligned}$$

The initial condition is $c(0) = 0.04$, $V(0) = 3$ so $a(0) = 0.12$. So

$$0.12 = a(0) = \frac{C}{3} \Rightarrow C = 0.36.$$

Hence the solution is

$$a(t) = \frac{0.36}{3+t}.$$

We wish to find the value of t for which

$$c(t) = \frac{a(t)}{V(t)} = \frac{0.36}{(3+t)^2} = 0.04,$$

or

$$\begin{aligned}(3+t)^2 &= 0.36/0.04 = 9 \\ 3+t &= 3,\end{aligned}$$

that is, $t = 3$.
