

# Math 211

## Ordinary Differential Equations and Linear Algebra Solutions for First Midterm Exam, Spring 2002

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1. Suppose you are given the differential equation  $(t - 3)^2 y' = 3y$ .

(a) (7 points) Is the function  $y(t) = \frac{1}{(t-3)}$  a solution?

*Solution:* We need to see if the function makes the ODE true. To do this we calculate LHS and RHS of the ODE using the function  $y(t) = \frac{1}{(t-3)}$ .

$$\text{LHS} = (t - 3)^2 y' = (t - 3)^2 \left( \frac{d}{dt} \left( \frac{1}{(t-3)} \right) \right) = (t - 3)^2 \left( \frac{-1}{(t-3)^2} \right) = -1.$$

$$\text{RHS} = 3y = \frac{3}{t-3}$$

Since  $\text{LHS} \neq \text{RHS}$  the function  $y(t) = \frac{1}{(t-3)}$  is NOT a solution of this ODE.

(b) (3 points) What is the value of the slope of the tangent line of a solution passing through the point  $(t, y) = (4, 1)$ ?

*Solution:* The slope of the tangent line of a solution is the value of the derivative of the solution. Thus if  $y = y(t)$  is our solution we need to calculate  $y'(4, 1)$ . But the ODE says that

$$y' = \frac{3y}{(t-3)^2}$$

and so

$$y'(4, 1) = \frac{3(1)}{(4-3)^2} = 3$$

Thus the answer is 3.

(c) (15 points) Solve the given differential equation with initial condition  $y(0) = 3e$  including the interval of existence (here  $e$  is the base of the natural logarithm i.e.  $e = e^1 = 2.71828\dots$ ).

*Solution:* This ODE is separable and can be separated to:

$$\frac{dy}{y} = \frac{3dt}{(t-3)^2}$$

Finding the antiderivative of both sides we get:

$$\ln|y| = \frac{-3}{t-3} + C, C = \text{constant}$$

Now solve for  $y$ :

$$|y| = e^{(-3/(t-3))+C}$$

Then

$$|y| = Ke^{-3/(t-3)}, K = e^C$$

Then  $y = Ke^{-3/(t-3)}$  if we let  $K = \pm e^C$ .

Now apply the initial condition  $y(0) = 3e$  to get  $3e = Ke^{-3/(0-3)}$  i.e.  $K = 3$ .

Thus the formula for the solution is  $y = 3e^{-3/(t-3)}$ .

We also need the interval of existence. During the method we divide by both  $y$  and  $t - 3$  so we need both these quantities to be non-zero. But for our solution,  $y > 0$  for any value of  $t$  so this is not an issue. However we need  $t \neq 3$  (in fact the solution is discontinuous at this  $t$ -value also). So our possible intervals are  $t < 3$  or  $t > 3$ . Since our initial condition is at  $t = 0$  we must choose the interval  $t < 3$ .

So the final solution is:  $y = e^{-3/(t-3)}, t < 3$ .

2. Suppose a mass of 4kg is moving under the influence of gravity (use  $g = 10 \text{ m/s}^2$ ).

- (a) (10 points) If the air-resistance is proportional to the velocity of the object with constant of proportionality  $r$  (where  $r > 0$ ), find the formula for the velocity of the object for  $t \geq 0$  if the velocity at time  $t = 0$  seconds is  $\frac{20}{r} \text{ m/s}$  (NOTE: this formula will involve  $r$ ).

*Solution:* Use Newton's Law to model this motion. Let  $v = v(t)$  = velocity of the object (towards ground is negative) in  $m/s$  at time  $t$  in seconds. Then Newton's law says: (mass)x(acceleration of object)=forces acting on it. There is the force of gravity acting downwards and there is air resistance acting in the opposite direction to velocity. So the equation is:

$$m \frac{dv}{dt} = -mg - rv$$

Using  $m = 4\text{kg}$  and  $g = 10\text{m/s}^2$  and simplifying we get:

$$\frac{dv}{dt} = -10 - rv/4$$

We also have the initial condition:  $v(0) = 20/r$ .

Now solve this ODE for  $v$ : separate to get  $\frac{dv}{10+rv/4} = -dt$ . Finding antiderivative of both sides gives:

$$(4/r)\ln|10 + rv/4| = -t + C, C = \text{constant}$$

Now solve for  $v$ :

$$|10 + rv/4| = Ke^{-rt/4}, K = e^{rC/4}$$

Letting  $K = \pm e^{rC/4}$  gives

$$10 + rv/4 = Ke^{-rt/4}$$

and solving gives finally

$$v = v(t) = (4K/r)e^{-rt/4} - 40/r$$

Using the initial condition  $v(0) = 20/r$  gives  $K = 15$  and so the equation is

$$v = v(t) = (60/r)e^{-rt/4} - 40/r$$

- (b) (5 points) Suppose you also know that the terminal velocity of the object is -10 m/s. Find the time at which the object is at its highest point.

*Solution:* Need to find time  $t$  when  $v(t) = 0$ . To do this we need to know the value of  $r$ .

When air-resistance is proportional to velocity we know that  $v_{term} = -mg/r = -40/r$ . Alternatively, using our solution above,  $v_{term} = \lim_{t \rightarrow +\infty} v(t) = -40/r$ . If  $v_{term} = -10/r$  this gives  $r = 4$ . Hence the actual velocity function for this object is

$$v = v(t) = 15e^{-t} - 10$$

Now solve  $v(t) = 0$  i.e.  $0 = 15e^{-t} - 10$  giving  $t = -\ln(2/3) = \ln(3/2)$ .

- (c) (5 points) Find the height of the object at the highest point if it starts out at a height of 15 metres above ground level (do not simplify your expression for the height).

*Solution:* Let  $x(t)$  =height (in metres) of object above ground at time  $t$  in seconds. Need to find  $x(\ln(3/2))$ . To do this we need to find the formula for  $x = x(t)$ . But

$$\frac{dx}{dt} = v(t) = 15e^{-t} - 10$$

Also we have the initial condition  $x(0) = 15$ . Solving this gives

$$x(t) = -15e^{-t} - 10t + 30$$

Then the required solution is

$$x(\ln(3/2)) = -15e^{-\ln(3/2)} - 10(\ln(3/2)) + 30$$

3. (20 points) Suppose a swimming pool of total volume 4000 litres initially has 1000 litres of a salt solution of concentration 2 kg/litre. If salt solution of concentration 0.4 kg/litre is being poured into the pool at a rate of 10 litres/minute and solution is leaving the pool at a rate of 5 litres/minute, find the total amount of salt in the pool at the time it becomes full (assume instantaneous mixing of all solutions).

- (a) (5 points) Show that the differential equation modelling the total amount of salt,  $A = A(t)$  (measured in kg) in the pool at time  $t$  (minutes) is given by:

$$\frac{dA}{dt} = 4 - A/(200 + t)$$

*Solution:* Let  $A(t)$  =total amount (in kg) of salt in the pool at time  $t$  (minutes). Let  $V(t)$  =volume of salt solution in the pool at time  $t$ .

We know that  $A(0) = (2)(1000) = 2000\text{kg}$ . Also  $V(0) = 1000$ .

Comparing the flow rates we see that there is a net INFLOW of 5litres/minute. Thus we have  $V(t) = 1000 + 5t$ .

Our model is:  $\frac{dA}{dt} = (\text{rate of salt coming in}) - (\text{rate of salt leaving})$ .  
 Note the units of this equation are kg/minute.

rate of salt coming in = (flow rate in) × (concentration coming in)

Thus rate of salt coming in = (10)(0.4) = 4kg/minute.

Similarly rate of salt leaving = (5)(A(t)/V(t)) = (5)(A/(1000+5t)) = A/(200+t)

Thus our model is:

$$\frac{dA}{dt} = 4 - A/(200 + t)$$

- (b) (12 points) Solve this differential equation to find a formula for  $A = A(t)$ .

*Solution:* This is a linear ODE with  $a = a(t) = -1/(200+t)$  and  $f = f(t) = 4$ .  
 Solve this using either method:

Integrating factor method:  $u = e^{-\int a dt} = e^{\ln|200+t|} = |200+t|$ . Since  $t \geq 0$ ,  $200+t \geq 0$  so  $|200+t| = 200+t$ . Thus  $u = 200+t$ .

Then  $A = A(t) = \frac{1}{u} \int u f dt + C/u$ ,  $C = \text{constant}$ .

Solving gives  $A = A(t) = 400 + 2t + C/(200+t)$ . Using  $A(0) = 2000$  gives  $C = 320000$ .

Variation of parameters:  $y_h = y_h(t)$  solves  $\frac{dA}{dt} = -A/(200+t)$ .  
 Solving gives  $y_h(t) = 1/(200+t)$ .

Then  $v = \int f/y_h dt + C = \int 4(200+t) dt + C = 2(200+t)^2 + C$ ,  $C = \text{constant}$ .

Then the solution is  $A = A(t) = v y_h = 400 + 2t + C/(200+t)$ . Using  $A(0) = 2000$  gives  $C = 320000$ .

Thus  $A(t) = 400 + 2t + 320000/(200+t)$  using either method.

- (c) (3 points) Find the total amount of salt in the pool at the time it becomes full.

*Solution:* The pool is full when  $V(t) = 4000$ . So we need to solve  $4000 = 1000 + 5t$ . This gives  $t = 600$  minutes. Thus we are trying to find  $A(600)$ .

Using our formula from part (b) we get  $A(600) = 2000$  (exactly the same value as we started with!).

4. Suppose a lake has a logistic model for the population of fish  $P = P(t)$  ( $t = \text{time in days}$ ) given by the equation  $\frac{dP}{dt} = P(1 - P/600)$ . Suppose also that people fish this lake and remove 10 percent of the fish population per day.

- (a) (2 points) Write a model for the population of fish in the lake.

*Solution:*  $\frac{dP}{dt} = P(1 - P/600) - 0.1P$ . Simplifying this gives

$$\frac{dP}{dt} = P(0.9 - P/600)$$

- (b) (10 points) Find the phase-line for this model.

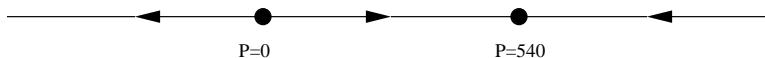
*Solution:* First locate the equilibrium solutions. These occur when RHS=0 i.e. solve  $0 = P(0.9 - P/600)$  giving  $P = 0$  or  $P = 540$ .

Now for each of the three regions ( $P < 0$ ,  $0 < P < 540$  and  $P > 540$ ) we need to calculate the sign of the derivative of solutions in these regions. To do this we use the differential equation.

When  $P < 0$ ,  $P(0.9 - P/600) < 0$  hence  $\frac{dP}{dt} = P(0.9 - P/600) < 0$ . Thus the solutions are decreasing in this region.

Similarly when  $0 < P < 540$ ,  $\frac{dP}{dt} > 0$  so solutions are increasing in this region.

When  $P > 540$ ,  $\frac{dP}{dt} < 0$  so solutions are decreasing in this region. Thus the phaseline looks like:



(c) (4 points) Classify each equilibrium solution as stable or unstable.

*Solution:* For the equilibrium solution  $P = 0$ , solutions either side head away from  $P = 0$  as time increases so this is an UNSTABLE equilibrium.

For the equilibrium solution  $P = 540$ , solutions either side head towards  $P = 540$  as time increases so this is a STABLE equilibrium.

(d) (4 points) If at  $t = 0$  the population is 200 fish, what value will the population of fish approach as  $t \rightarrow +\infty$ ?

*Solution:* If  $P(0) = 200$  this puts us in the region  $0 < P < 540$ . Hence the solution will increase and in fact increases towards the stable equilibrium solution  $P = 540$ . Hence the population of fish starting at 200 will approach the value 540 as  $t \rightarrow +\infty$ .

5. Suppose you are given the differential equation  $y' = -e^{2t}y^3$ .

(a) (5 points) Find the values of  $K$  that make  $y(t) = Ke^{-t}$  a solution to this equation (where  $K$  is a constant).

*Solution:* For  $y(t) = Ke^{-t}$  to be a solution we need the LHS and RHS of the ODE to be equal when calculated using  $Ke^{-t}$ . So we calculate LHS and RHS.

$$\text{LHS} = y' = \frac{d}{dt}(Ke^{-t}) = -Ke^{-t}.$$

$$\text{RHS} = -e^{2t}y^3 = -e^{2t}(Ke^{-t})^3 = -K^3e^{-t}.$$

Hence if LHS=RHS we need  $-Ke^{-t} = K^3e^{-t}$  i.e. we need  $-K = -K^3$ . Solving this gives  $K = 0, \pm 1$ .

(b) (10 points) If  $y = y(t)$  is also a solution and  $y(0) = 0.5$  what is  $\lim_{t \rightarrow +\infty} y(t)$ ? (NOTE: you don't necessarily need to solve the ODE to answer this question but you must justify your answer fully).

*Solution:* From part (a) we know that  $y(t) = e^{-t}$  and  $y(t) = 0$  are solutions to the ODE. In particular at  $t = 0$  they have values 1 and 0 respectively. The solution we are looking at starts at  $y(0) = 0.5$  i.e it starts between these two solutions. If we knew we had unique solutions for all points in the solution plane with  $t \geq 0$  then the solution we are looking at would be forced to stay between  $y(t) = e^{-t}$  and  $y = 0$  as  $t \rightarrow +\infty$ . And since  $\lim_{t \rightarrow +\infty} e^{-t} = 0$  this would mean that the solution we are interested in has limit 0 as  $t \rightarrow +\infty$ .

Thus we will be done if we know we have unique solutions at least for all times  $t \geq 0$ . We check that this is the case. We need to show that  $-e^{2t}y^3$  and  $\frac{\partial}{\partial y}(-e^{2t}y^3)$  are continuous for  $t \geq 0$ .

In fact  $-e^{2t}y^3$  is continuous at any point  $(t, y)$  and hence for any point with  $t \geq 0$ .

Calculating  $\frac{\partial}{\partial y}(-e^{2t}y^3) = -e^{2t}(3y^2)$  and this is also continuous at any point  $(t, y)$  and hence for any point with  $t \geq 0$ .

Thus we know that the ODE has unique solutions for  $t \geq 0$  and so we know that  $\lim_{t \rightarrow +\infty} y(t) = 0$  if  $y(0) = 0.5$ .